Lecture Notes on Optimal Spacecraft Guidance

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Preface

These notes are for the one semester course at Utah State University titled MAE 6570 Optimal Spacecraft Guidance. The class meets for 75 minutes, twice per week, for 14 weeks. Standard text is written in dark blue and examples are written in green. Points of emphasis are written in orange and warnings are written in red. Sections and subsections are highlighted in orange. Figures may include many colors.

Errors will be corrected when identified. New versions will be released at the end of each semester that the course is taught. If you identify errors, please email matthew.harris@usu.edu.

Prerequisites for the class include graduate standing. No textbook is required. Material is pulled from various sources on space dynamics, optimization, optimal control, and guidance.

1. Curtis, Orbital Mechanics for Engineering Students, 4th edition, 2020.
2. Berkovitz, Convexity and Optimization in $\mathbb{R}^n$, 1st edition, 2002.
3. Lewis and Syrmos, Optimal Control, 2nd edition, 1995.

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I thank Professor David Hull at the University of Texas and Professor Behçet Açıkmeşe at the University of Washington for first teaching me the subject. I thank all the students that have helped improve the class. Best of luck. – Matt Harris

Contents

Introduction to Guidance

Most vehicles (in air, space, or water) include guidance, navigation, & control (GN+C) systems that operate in real-time as the vehicle moves.

The navigation system consists of sensors to measure the state of the system as well as tools for filtering, outlier detection, estimation, etc.

The guidance system uses the current estimate of the state provided by the navigation system along with the mission objectives to compute state and control trajectories.

The control system uses the control trajectory provided by the guidance system to compute control commands to affect actuators (engines, wing surfaces, etc.)

Example: When you drive a car, you are part of all three systems. An app that tells you to "turn right in 25 ft" is issuing a guidance command.

A good GN&C system is one that is

- robust to measurement noise, uncertainties, disturbances, unmodeled dynamics, etc.
- stable so that small errors don't cause large changes in results.
- simple enough so that it can run in real-time.

Such a system is demonstrated in the YouTube video:

"Apollo 12 landing from PDI to Touchdown"

The three components of GN&C are obviously coupled.

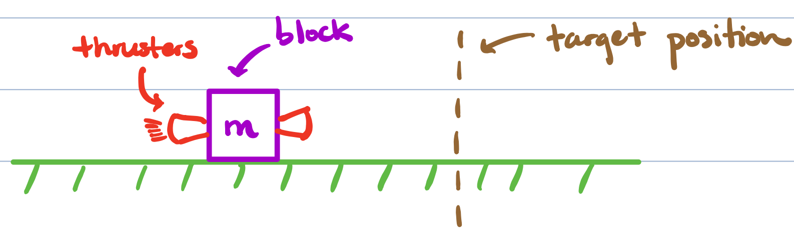
- The guidance system should not rely on state estimates unavailable from the navigation system.
- The guidance system should not generate control commands beyond the actuator limits.
- And so on...

The guidance system may generate trajectories by:

- using a reference trajectory (or a priori plan).
- solving an optimization problem.
- interpolation, approximation, or other means.

In any case, the method must be quick & guaranteed to work. (Imagine a rocket crashing because the Newton solver fails to converge.)

Example: Let's consider a block that can slide in 1-D along a line. It can choose to thrust left or right (but not both). The goal is for the block to pass through the target position.



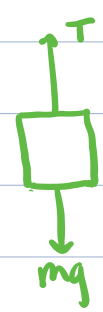
Because the block doesn't have to stop at the target position, it is clear the block should:

- thrust left (move right) when left of the target
- thrust right (move left) when right of the target.

We just solved our first guidance problem! Note that we had to know our position (from navigation) and the target position (mission objective).

How would our solution change if we had to pass through the target at a certain time? What if we had to stop? We would then need to know position, velocity, dynamic model, and control limits.

Example: A lunar lander is 10 m above its landing site and has 1 m/s downward velocity. Its goal is to descend to the surface and touch down with 1 m/s velocity. What thrust acceleration should be applied?



$$m\ddot{h} = T - mg$$
$$\Rightarrow \ddot{h} = T/m - g$$
$$= a_t - g$$

To descend at constant velocity, $\ddot{h} = a_t - g = 0$.
Hence, $a_t = g$. Will the thrust force be constant?
What role do navigation, mission objectives, & dynamics play in our solution?

Let t_0 be the current time and t_f be the final time. The problem of finding a function

$$a_t : [t_0, t_f] \longrightarrow \mathbb{R}^3$$

that minimizes some objective is called an optimal control problem. In this landing problem, we may want to minimize

- fuel consumed
- time to touch down
- accelerations felt by the crew
- etc.

The problem of finding a guidance solution that also minimizes some quantity is an optimal guidance problem. The solution is called an optimal guidance law. To solve such problems, we will need to understand the:

- dynamical model } and reasonable assumptions
- control model
- optimality conditions
- numerical techniques

Dynamical Models

Our guidance algorithms depend on the dynamics. We will need a dynamical model that is simple enough and accurate enough. In contrast, simulation models are higher fidelity and may include many small perturbations (gravitational, atmospheric, etc.).

Perhaps the most important equation in spaceflight is the two-body equation of motion.

gravitational param. — initial time — initial velocity

2nd time derivative →

$$\ddot{\bar{r}} = -\frac{\mu}{r^3}\bar{r} \qquad \bar{r}(t_0) = \bar{r}_0, \quad \bar{v}(t_0) = \bar{v}_0$$

initial position

Solutions are the standard circles, ellipses, parabolas, and hyperbolas (depending on the initial conditions).

If we include thrust acceleration and disturbance accelerations, the equation becomes

$$\ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} = \bar{a}_t + \bar{a}_d, \quad \bar{r}(t_0) = \bar{r}_0, \quad \bar{v}(t_0) = \bar{v}_0$$

disturbance

thrust

We may write this equation in state-space form as

$$\begin{pmatrix} \dot{\bar{r}} \\ \dot{\bar{v}} \end{pmatrix} = \begin{pmatrix} \bar{v} \\ -\mu/r^3\,\bar{r} + \bar{a}_t + \bar{a}_d \end{pmatrix}$$

Because $\bar{a}_t$ is our "control" variable, it is common to rewrite as

$$\begin{pmatrix} \dot{\bar{r}} \\ \dot{\bar{v}} \end{pmatrix} = \begin{pmatrix} \bar{v} \\ -\mu/r^3\,\bar{r} + \bar{a}_d \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{a}_t \end{pmatrix}$$

With given initial conditions $\bar{r}_0$ and $\bar{v}_0$, known $\bar{a}_t$ at each time, and known $\bar{a}_d$ at each time, the nonlinear dynamical system can be integrated. For much of the course, this will serve as our simulation model.

Some of our guidance models can be attained by making simplifying assumptions.

A Planetary Powered Descent Model

We will assume that the vehicle is sufficiently close to the surface that gravity is constant (i.e., a flat planet model). We will further assume that the thruster acceleration dominates any disturbances.

$$\dot{\bar{r}} = \bar{v} \qquad , \qquad \bar{r}(t_0) = \bar{r}_0$$
$$\dot{\bar{v}} = \bar{g} + \bar{a}_t \qquad \bar{v}(t_0) = \bar{v}_0$$

Example: Let's assume that $\bar{a}_t$ is constant. Then, integrating is simple.

$$\bar{v} = \bar{g}t + \bar{a}_t t + \bar{v}_0$$

$$\bar{r} = \frac{1}{2}\bar{g}t^2 + \frac{1}{2}\bar{a}_t t^2 + \bar{v}_0 t + \bar{r}_0$$

Suppose that we want $\bar{r}(t_f) = \bar{0}$. Then, the required thrust acceleration is

$$\bar{a}_t = \frac{-2}{t_f^2}\left[\frac{1}{2}\bar{g}t_f^2 + \bar{v}_0 t_f + \bar{r}_0\right]$$

What happens as t_f (also called the time-to-go) approaches zero? Within any guidance algorithm, care must be taken when $t_f \to 0$.

How does this simple guidance law perform in simulation?

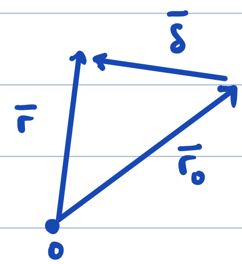
Are there enough degrees of freedom to also hit a desired velocity target? How could we add degrees of freedom?

In the Apollo lunar landing guidance, they used the above dynamical model (with constant gravity) and assumed a quadratic thrust acceleration, which leads to a quartic position trajectory.

Relative Orbital Motion

We'll now consider two spacecraft near each other in orbit. We will derive, from the two-body equation, equations describing the relative motion that are linear. As with the descent model above, linearity is nice because it facilitates integration.

Let the target spacecraft position be $\bar{r}_0$ and the chaser spacecraft be $\bar{r}$. The relative position is $\bar{\delta}$, i.e., $\bar{r} = \bar{r}_0 + \bar{\delta}$.



The equation of motion for the chaser is

$$\ddot{\bar{r}} = -\frac{\mu}{r^3}\bar{r} + \bar{a}_t$$

$$\Rightarrow \quad \ddot{\bar{\delta}} = -\ddot{\bar{r}}_0 - \frac{\mu}{r^3}\left(\bar{r}_0 + \bar{\delta}\right) + \bar{a}_t$$

Observe that

$$r^2 = \bar{r} \cdot \bar{r} = (\bar{r}_0 + \bar{\delta}) \cdot (\bar{r}_0 + \bar{\delta})$$

$$= \bar{r}_0 \cdot \bar{r}_0 + 2\bar{r}_0 \cdot \bar{\delta} + \bar{\delta} \cdot \bar{\delta}$$

$$= r_0^2 + 2\bar{r}_0 \cdot \bar{\delta} + \delta^2 = r_0^2 \left[1 + \frac{2\bar{r}_0 \cdot \bar{\delta}}{r_0^2} + \left(\frac{\delta}{r_0} \right)^2 \right]$$

We now assume that $\delta / r_0 \ll 1$ such that the last term can be neglected.

$$\Rightarrow r^2 \approx r_0^2 \left[1 + \frac{2\bar{r}_0 \cdot \bar{\delta}}{r_0^2} \right]$$

$$\Rightarrow r^{-3} \approx r_0^{-3} \left[1 + \frac{2\bar{r}_0 \cdot \bar{\delta}}{r_0^2} \right]^{-3/2}$$

Expanding using the binomial theorem and keeping only 1st-order terms in $\bar{\delta}$ yields

$$r^{-3} \approx r_0^{-3} \left[1 - \frac{3}{r_0^2} \bar{r}_0 \cdot \bar{\delta} \right] = \frac{1}{r_0^3} - \frac{3}{r_0^5} \bar{r}_0 \cdot \bar{\delta}$$

Substituting into our $\ddot{\bar{\delta}}$ equation gives

$$\ddot{\bar{\delta}} \approx -\ddot{\bar{r}}_0 - \mu\left[\frac{1}{r_0^3} - \frac{3}{r_0^5}\bar{r}_0\cdot\bar{\delta}\right](\bar{r}_0 + \bar{\delta}) + \bar{a}_t$$

Expanding and keeping only 1^{st}-order terms gives

$$\ddot{\bar{\delta}} \approx -\ddot{\bar{r}}_0 - \mu\frac{\bar{r}_0}{r_0^3} - \frac{\mu}{r_0^3}\left[\bar{\delta} - \frac{3}{r_0^2}(\bar{r}_0\cdot\bar{\delta})\bar{r}_0\right] + \bar{a}_t$$

Assume that $\ddot{\bar{r}}_0 = {}^{-\mu}/_{r_0^3}\,\bar{r}_0$. Therefore,

$$\ddot{\bar{\delta}} \approx -\frac{\mu}{r_0^3}\left[\bar{\delta} - \frac{3}{r_0^2}(\bar{r}_0\cdot\bar{\delta})\bar{r}_0\right] + \bar{a}_t$$

Since the equation is linear, it can be written in state-space form.

$$\begin{pmatrix} \dot{\bar{\delta}} \\ \dot{\bar{v}} \end{pmatrix} = \left(\begin{array}{c|c} 0 & I \\ -{}^{\mu}/_{r_0^3}\left(I - {}^{3}/_{r_0^2}\,\bar{r}_0\bar{r}_0^T\right) & 0 \end{array}\right)\begin{pmatrix} \bar{\delta} \\ \bar{v} \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{a}_t \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \dot{\bar{\delta}} \\ \dot{\bar{v}} \end{pmatrix} = A(\bar{r}_0(t))\begin{pmatrix} \bar{\delta} \\ \bar{v} \end{pmatrix} + B\,\bar{a}_t$$

We have written these equations in a frame independent fashion. If we assume that the target spacecraft moves in a circular orbit and if we attach a local vertical local horizontal (LVLH) frame to the target, we will obtain a very special case known as the Clohessy-Wiltshire (CW) equations.

The local vertical direction is defined as $\hat{\imath} = \bar{r}_0 / r_0$, and the local vertical position & velocity are x and $\dot{x}$.

The out-of-plane direction is defined as $\hat{k} = \dfrac{\bar{r}_0 \times \bar{v}_0}{\|\bar{r}_0 \times \bar{v}_0\|}$. Coordinates are z and $\dot{z}$.

The local horizontal direction is $\hat{\jmath}$ where $\hat{\imath} \times \hat{\jmath} = \hat{k}$. Coordinates are y & $\dot{y}$.

Given a $\bar{\delta}$ in an inertial frame, we can transform to the LVLH frame using

$$x = \bar{\delta} \cdot \hat{\imath}, \quad y = \bar{\delta} \cdot \hat{\jmath}, \quad z = \bar{\delta} \cdot \hat{k}$$

The velocity transformation must account for the fact that the velocity frame is rotating with the target spacecraft.

$$\dot{x} = (\bar{v} - \bar{\omega} \times \bar{\delta}) \cdot \hat{i}$$
$$\dot{y} = (\bar{v} - \bar{\omega} \times \bar{\delta}) \cdot \hat{j}$$
$$\dot{z} = (\bar{v} - \bar{\omega} \times \bar{\delta}) \cdot \hat{k}$$

The CW equations are then

$$\begin{aligned} \ddot{x} - 3n^2 x - 2n\dot{y} &= a_x \\ \ddot{y} + 2n\dot{x} &= a_y \\ \ddot{z} + n^2 z &= a_z \end{aligned} \quad , \qquad n = \sqrt{\mu / r_0^3}$$

We make several observations about the CW equations.

1) The equations are linear and time-invariant.
2) The out-of-plane motion is a decoupled harmonic oscillator.
3) The in-plane motion is coupled.
4) The control appears linearly.

In the case where $a_x = a_y = a_z = 0$, the CW equations can be integrated analytically.

$$x = (4 - 3\cos nt)\,x_0 + \frac{\dot{x}_0}{n}\sin nt + \frac{2}{n}(1 - \cos nt)\,\dot{y}_0$$

$$y = 6(\sin nt - nt)\,x_0 + y_0 + \frac{2}{n}(\cos nt - 1)\,\dot{x}_0 + \frac{1}{n}(4\sin nt - 3nt)\,\dot{y}_0$$

$$z = z_0 \cos nt + \frac{\dot{z}_0}{n}\sin nt$$

$$\dot{x} = 3n x_0 \sin nt + \dot{x}_0 \cos nt + 2\dot{y}_0 \sin nt$$

$$\dot{y} = 6n(\cos nt - 1)\,x_0 - 2\dot{x}_0 \sin nt + (4\cos nt - 3)\,\dot{y}_0$$

$$\dot{z} = -n z_0 \sin nt + \dot{z}_0 \cos nt$$

By factoring out the initial conditions, we can write the equations in matrix form.

$$\left[\begin{array}{c} x \\ y \\ z \\ \hline \dot{x} \\ \dot{y} \\ \dot{z} \end{array}\right] = \left[\begin{array}{c|c} \Phi_{rr}(t) & \Phi_{rv}(t) \\ \hline \Phi_{vr}(t) & \Phi_{vv}(t) \end{array}\right] \left[\begin{array}{c} x_0 \\ y_0 \\ z_0 \\ \hline \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{array}\right]$$

Each of the submatrices is 3×3. The matrix

$$\Phi(t) = \begin{bmatrix} \Phi_{rr}(t) & \Phi_{rv}(t) \\ \Phi_{vr}(t) & \Phi_{vv}(t) \end{bmatrix}$$

is called the state transition matrix.

Properties of State Transition matrices

As we've discussed, linear dynamical systems may be written in the standard state-space form:

$$\dot{\bar{x}}(t) = \overset{\text{system matrix}}{A(t)}\,\underset{\text{state}}{\bar{x}(t)} + \overset{\text{control influence matrix}}{B(t)}\,\underset{\text{control}}{\bar{u}(t)}$$

Theorem: Suppose zero-input and A is continuous. For any $t_0, \bar{x}_0$ there is a unique continuously differentiable solution

$$\bar{x}(t) = \Phi(t, t_0)\,\bar{x}_0 .$$

We call Φ the fundamental matrix or state transition matrix (STM). $\blacksquare$

The STM has a series definition you can look up. It also satisfies some interesting properties.

- $\frac{d}{dt}\Phi(t,t_0) = A(t)\,\Phi(t,t_0)\,, \quad \Phi(t_0,t_0) = I$
- $\Phi(t,t_0) = \Phi^{-1}(t_0,t)$
- $\Phi_{-A^T}(t,t_0) = \Phi_A^{-T}(t,t_0) = \Phi_A^{T}(t_0,t)$
- $\Phi(t_2,t_0) = \Phi(t_2,t_1)\,\Phi(t_1,t_0)$

Solutions to Forced Linear Systems

The general solution to the forced (controlled) linear system is given by

$$\bar{x}(t) = \Phi(t,t_0)\,\bar{x}_0 + \int_{t_0}^{t} \Phi(t,\sigma)\,B(\sigma)\,\bar{u}(\sigma)\,d\sigma$$

You can verify it is a solution by differentiation.

Discretization of Linear Systems

To this point, we've discussed continuous time systems because these naturally arise in physics. However, the nature of guidance is discrete because we call the system at some frequency.

Suppose we discretize time as

$$t_0 < \dots < t_i < t_{i+1} < \dots < t_f$$

and we hold the control constant on every interval. Then,

$$\bar{x}(t_{i+1}) = \underbrace{\Phi(t_{i+1}, t_i)}_{A_i} \bar{x}(t_i) + \underbrace{\int_{t_i}^{t_{i+1}} \Phi(t_{i+1}, \sigma) B(\sigma) \, d\sigma}_{B_i} \bar{u}(t_i)$$

$$\Rightarrow \quad \bar{x}_{i+1} = A_i \bar{x}_i + B_i \bar{u}_i$$

By using the state transition matrix, we can convert continuous time systems to discrete time systems. We will discuss nonlinear systems later.

Mass Dynamics

We've been thinking of our states as positions and velocities. Another key state is mass, m. Throughout the course, we will model the mass dynamics as

$$\dot{m} = \frac{-1}{I_{sp} g_0} \|\bar{T}\|$$

where g_0 is the standard sea-level acceleration of gravity on Earth and I_{sp} is the engine's specific impulse (in sec). The magnitude of the thrust force is $\|\bar{T}\|$.

Note that the equation is nonlinear in the thrust vector because

$$\|\bar{T}\| = \left[T_x^2 + T_y^2 + T_z^2 \right]^{1/2}.$$

Moreover, because $\bar{a}_t = \bar{T}/m$, we almost always have to deal with nonlinear dynamics.

Example: Solid rockets provide a constant thrust. In this case, the mass varies linearly with time.

$$m = m_0 - \frac{\|T\|}{g_0 I_{sp}} t$$

This is useful because the $\dot{m}$ equation can be eliminated and the $\bar{T}/m$ terms now appear linearly.

Introduction to Optimization

We previously discretized a linear dynamical system using the state transition matrix and assuming piecewise constant controls.

$$\bar{x}_{i+1} = A\bar{x}_i + B\bar{u}_i$$

By writing out a few terms we can see how the final state depends on the initial state and control inputs.

$$\bar{x}_1 = A\bar{x}_0 + B\bar{u}_0$$

$$\begin{aligned} \bar{x}_2 &= A\bar{x}_1 + B\bar{u}_1 \\ &= A^2\bar{x}_0 + AB\bar{u}_0 + B\bar{u}_1 \end{aligned}$$

$$\begin{aligned} \bar{x}_3 &= A\bar{x}_2 + B\bar{u}_2 \\ &= A^3\bar{x}_0 + A^2B\bar{u}_0 + AB\bar{u}_1 + B\bar{u}_2 \\ &\vdots \\ \bar{x}_N &= A^N\bar{x}_0 + \sum_{i=0}^{N-1} A^{(N-1-i)} B\bar{u}_i \end{aligned}$$

By stacking all of the controls into a tall $(Nm \times 1)$ vector

$$\mathcal{U} = \begin{bmatrix} \bar{u}_0 \\ \bar{u}_1 \\ \vdots \\ \bar{u}_{N-1} \end{bmatrix}$$

and defining the matrix

$$\mathcal{C} = \underbrace{\left[A^{N-1}B,\ A^{N-2}B,\ \dots,\ AB,\ B \right]}_{n \times Nm}$$

the above equation can be written as

$$\bar{x}_N - A^N \bar{x}_0 = \mathcal{C}\,\mathcal{U}$$

Note that $\mathcal{C}$ is called the "controllability matrix." If the system is controllable, then $\operatorname{im}(\mathcal{C}) = \mathbb{R}^n$ and the linear system is solvable.

Suppose that matrix $\mathcal{C}$ has a non-trivial nullspace. If there is one solution to the equation, then there are infinitely many solutions. Let u_p be some particular solution to the equation and let L be a matrix such that

$$\mathrm{im}(L) = \mathrm{null}(\mathcal{C}).$$

Then all solutions are given by

$$u = u_p + LV$$

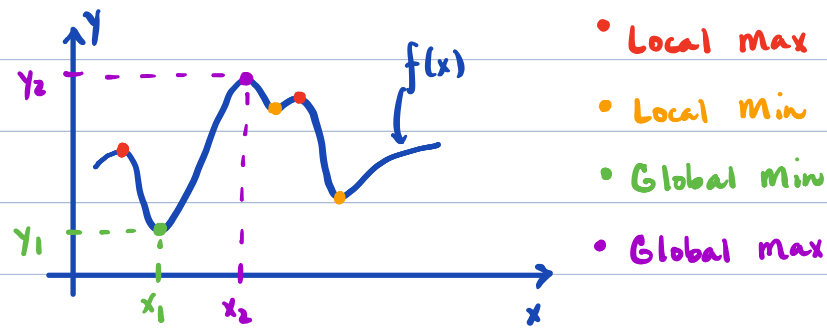
for any V. If there are infinitely many control trajectories to drive $\bar{x}_0$ to $\bar{x}_N$, how do we choose one?

Optimize some objective!

Common objectives include fuel, energy, and time. By fixing the 0 and N indices we cannot minimize time. We'll save that for later and focus for now on some basics + fuel/energy objectives.

A Few Basics

To optimize means to minimize or maximize. We already have an intuitive (graphical) understanding of the concept.



Mathematically, we say

$$x_1 \in \operatorname{argmin} f \quad , \quad y_1 = \min f$$

$$x_2 \in \operatorname{argmax} f \quad , \quad y_2 = \max f$$

"argument that maximizes"

"maximum value"

These words have global meaning - not local.

We will always be interested in global optimization in this class, but we must watch out for local optima.

By drawing a few pictures, you can convince yourself of the following facts.

$$\operatorname{argmin} f = \operatorname{argmax} -f$$

$$\min f = -\max -f$$

Therefore, we will need a theory only for minimization problems.

From calculus, you may recall the following theorem.

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be smooth. If $x \in \operatorname{argmin} f$ then $f'(x) = 0$.

Any points that satisfy $f'(x) = 0$ are called critical points or candidates. Any optimal point must be a candidate, but not all candidates must be optimal points.

Question: What if there is only one candidate?

Example 1: minimize $f(x) = x^2$. $f'(x) = 2x = 0 \Rightarrow x = 0$.
We have one candidate that globally minimizes x^2.

Example 2: minimize $f(x) = x^3$. $f'(x) = 3x^2 = 0 \Rightarrow x = 0$
We have one candidate that does not minimize (locally or globally) x^3.

Example 3: minimize $f(x) = e^{-x}$. $f'(x) = -e^{-x} \neq 0$
There are no candidates and hence no minima.

Example 4: minimize $f(x) = (x-2)^2(x+2)^2$. There are 3 candidates $-2, 0, +2$. Two of them are global minima.

Example 5: minimize $f(x) = \sin(x)$. There are countably infinite candidates. There are countably infinite global minima & maxima.

Example 6: minimize $f(x) = 1$. There are uncountably many global minima that are also global maxima.

The six examples above, which involve single-variable analytic functions, demonstrate that just about anything can happen in optimization.

Problems with Constraints

All the problems we'll be interested in will have constraints (physics, thrust limits, boundary conditions, etc.). Therefore, we will now consider nonlinear programming problems (NLPs).

min	$f(x)$	objective	$f: \mathbb{R}^n \to \mathbb{R}$
s.t.	$g(x) \leq 0$	inequality constraints	$g: \mathbb{R}^n \to \mathbb{R}^p$
	$h(x) = 0$	equality constraints	$h: \mathbb{R}^n \to \mathbb{R}^q$

Note that g & h may be multi-valued but I am not putting a bar on them. Nor am I putting a bar on x. Just about everything is multi-dimensional from here on.

We define the constraint set to be

$$\mathcal{X} = \{ x \in \mathbb{R}^n : g(x) \leq 0, \ h(x) = 0 \}$$

We may now write $\min_{x \in \mathcal{X}} f$ and $x \in \operatorname{argmin}_{x \in \mathcal{X}} f$.

There are two types of necessary conditions for these problems:

1. Conditions with a regularity or constraint qualification are called KKT conditions.
2. Conditions without the qualification are called Fritz John conditions.

Many books focus entirely on KKT conditions. I prefer working with the FJ conditions since they don't require an additional qualification. We'll write down both & then solve some problems.

Theorem (KKT conditions for NLP): Assume that $f, g,$ & h are differentiable. If the problem attains a minimum at x and a constraint qualification holds, then the following system is solvable:

$$\begin{aligned}
g(x) &\preceq 0 \\
h(x) &= 0 \\
\lambda &\geq 0 \\
\lambda^T g(x) &= 0 \\
\nabla_x f(x) + \nabla_x g(x) \lambda + \nabla_x h(x) v &= 0 \qquad \square
\end{aligned}$$

There are numerous constraint qualifications that make the above theorem true. Two of the most common are:

1. Linear Independence CQ requires the gradients of all <u>active</u> constraints to be linearly independent at the optimal point.

2. Mangasarian-Fromowitz CQ requires the gradients of $h(x)$ to be linearly independent & the existence of a vector z s.t.

$$\nabla^T g_{\mathcal{A}}(x) z < 0, \quad \nabla^T h(x) z = 0$$

for all <u>active</u> constraints.

Two other CQs are the Abadie CQ & the Guignard CQ.

All of these CQs are evaluated at the optimal point. Thus, they cannot, in general, be verified a priori. For this reason, I prefer the FJ conditions.

Theorem (Fritz John Conditions for NLP): Assume that $f, g, \& h$ are differentiable. If the problem attains a minimum at x, then the following system is solvable:

$$
\begin{aligned}
& g(x) \le 0 \\
& h(x) = 0 \\
& (\lambda_0, \lambda, v) \neq 0, \ \lambda_0 \in \{0,1\}, \ \lambda \ge 0 \\
& \lambda^T g(x) = 0 \\
& \lambda_0 \nabla_x f(x) + \nabla_x g(x) \lambda + \nabla_x h(x) v = 0 \qquad \square
\end{aligned}
$$

Note that we now have a λ_0, which can only be zero or one. λ_0 is called the abnormal multiplier. A solution with $\lambda_0 = 1$ is a "normal solution." A solution with $\lambda_0 = 0$ is an "abnormal solution."

Don't let the lingo lead you astray. Abnormal solutions are quite common & should not be forgotten!

The condition that $(\lambda_0, \lambda, v) \neq 0$ is called the non-triviality condition.

Note that $\lambda^T g(x) = 0$ is called the complementarity condition. For every constraint, either $\lambda_i = 0$ or $g_i(x) = 0$.

Example: $\min \; x^2$
s.t. $(x-1)^2 = 0.$

It is obvious that $x=1$ is the answer since it is the only feasible point. Let's apply the FJ conditions.

$$L = \lambda_0 x^2 + \nu (x-1)^2$$
$$\frac{\partial L}{\partial x} = 2\lambda_0 x + 2\nu(x-1) = 0$$

Suppose $\lambda_0 = 1$. Then

$$2x(1+\nu) = 2\nu \Rightarrow x = \frac{\nu}{1+\nu}$$

To be feasible, x must equal 1. Therefore,

$$\nu = \nu + 1 \Rightarrow 0 = 1$$

Thus, there are no normal solutions. Suppose $\lambda_0 = 0$. Then

$$2\nu(x-1) = 0$$

The non-triviality condition requires $\nu \neq 0$. Thus, $x = 1$. The global minimum is an abnormal solution.

Example: minimize $x_1^2 + x_2^2 + x_1x_2 - 3x_1$

s.t. $x_1 \geq 0$

$x_2 \geq 0$

The Lagrangian is

$$L = \lambda_0\left(x_1^2 + x_2^2 + x_1x_2 - 3x_1\right) - \lambda_1 x_1 - \lambda_2 x_2$$

The derivative is

$$\frac{\partial L}{\partial x_1} = 2\lambda_0 x_1 + \lambda_0 x_2 - 3\lambda_0 - \lambda_1 = 0 \qquad (1)$$

$$\frac{\partial L}{\partial x_2} = 2\lambda_0 x_2 + \lambda_0 x_1 - \lambda_2 = 0 \qquad (2)$$

The complementarity conditions are

$$\lambda_1 x_1 = 0, \quad \lambda_2 x_2 = 0.$$

We have four cases to investigate.

1. Suppose $x_1 = x_2 = 0$. Eq. 2 $\Rightarrow$ $\lambda_2 = 0$.
   Eq. 1 $\Rightarrow$ $3\lambda_0 = -\lambda_1$.

If $\lambda_0 = 0$, then $\lambda_1 = 0$ violating the non-triviality condition.
If $\lambda_0 = 1$, then $\lambda_1 = -3 \ngeq 0$. Case 1 is ruled out.

(2) Suppose $\lambda_1 = \lambda_2 = 0$. The non-triviality condition tells us that $\lambda_0 = 1$.

$$\text{Eq. 1} \Rightarrow \quad 2x_1 + x_2 = 3$$
$$\text{Eq. 2} \Rightarrow \quad x_1 + 2x_2 = 0$$

Solving this linear system gives $x_1 = 2$, $x_2 = -1$. This point is not feasible. Case 2 is ruled out.

(3) Suppose $x_1 = 0$ and $\lambda_2 = 0$.

$$\text{Eq. 1} \Rightarrow \quad \lambda_0 x_2 - 3\lambda_0 - \lambda_1 = 0$$
$$\text{Eq. 2} \Rightarrow \quad 2\lambda_0 x_2 = 0 \quad \Rightarrow \lambda_0 = 0 \text{ or } x_2 = 0.$$

If $\lambda_0 = 0$, then Eq. 1 gives $\lambda_1 = 0$ violating non-triviality.
If $x_2 = 0$, then $\lambda_1 = -3 \not\geq 0$. Case 3 has been ruled out.

(4) Suppose $\lambda_1 = 0$ and $x_2 = 0$.

$$\text{Eq. 1} \Rightarrow \quad 2\lambda_0 x_1 - 3\lambda_0 = 0$$
$$\text{Eq. 2} \Rightarrow \quad \lambda_0 x_1 - \lambda_2 = 0.$$

If $\lambda_0 = 0$, Eq. 2 gives $\lambda_2 = 0$ violating non-triviality.
If $\lambda_0 = 1$, then $x_1 = 3/2 = \lambda_2 \geq 0$.

Case 4 yielded a candidate. The FJ conditions are only necessary. They are not sufficient. If there is a solution, we've found it.

An Existence Theorem

After all that work, it would be nice to have a conclusive answer.

Weierstrass Theorem: If f is continuous and $\mathcal{X}$ is compact (closed & bounded), then f attains a minimum (and maximum) on $\mathcal{X}$.

In the above example, the domain is not compact. We can make it so by adding the constraints

$$x_1 \leq \sigma \quad \text{and} \quad x_2 \leq \sigma$$

We can now say that $(x_1, x_2) = (3/2, 0)$ minimizes f for any $3/2 \leq \sigma < \infty$. This is almost what we need. Is there any additional logic we can apply to deduce optimality?

Consider the following problem:

$$\begin{aligned} \max \quad & (x-1)^2 \\ \text{s.t.} \quad & x \leq 2 \end{aligned} \quad \longrightarrow \quad \begin{aligned} \min \quad & -(x-1)^2 \\ \text{s.t.} \quad & x \leq 2 \end{aligned}$$

We can then use the necessary conditions.

$$L = -\lambda_0 (x-1)^2 + \lambda (x-2)$$

$$\frac{\partial L}{\partial x} = -2\lambda_0 (x-1) + \lambda = 0$$

If $\lambda_0 = 0$, then $\lambda = 0$ violating non-triviality. Thus, $\lambda_0 = 1$.

If $\lambda = 0$, then $x = 1 \leq 2$

If $x = 2$, then $\lambda = 2 \geq 0$.

Thus, there are two candidates. Neither candidate gets the job since $f(x) \to -\infty$ as $x \to -\infty$.

If a solution exists, the solution will be one of the candidates. If a solution does not exist, the candidates will not be solutions (of course)!

Back to the motivating Problem

Recall that we started with the system

$$\xi_{NO} = \mathcal{C}\mathcal{U}$$

which we said would have infinitely many solutions.
Let's now find the "minimum energy" solution.
The energy is given by

$$\sum_i \|\bar{u}_i\|_2^2 = \sum_i \bar{u}_i^T \bar{u}_i = \mathcal{U}^T\mathcal{U} = \|\mathcal{U}\|_2^2$$

Therefore,

$$\begin{aligned} \min \quad & \tfrac{1}{2}\|\mathcal{U}\|_2^2 = \tfrac{1}{2}\mathcal{U}^T\mathcal{U} \\ \text{s.t.} \quad & \xi_{NO} = \mathcal{C}\mathcal{U} \end{aligned}$$

We first form the Lagrangian

$$L = \frac{\lambda_0}{2}\mathcal{U}^T\mathcal{U} + \lambda^T(\mathcal{C}\mathcal{U} - \xi_{NO})$$

The gradient is

$$\nabla_u L = \lambda_0 \mathcal{U} + \mathcal{C}^T\lambda = 0$$

Suppose $\lambda_0 = 0$. If the system is controllable, then $\mathcal{C}^T$ is full column rank and $\lambda = 0$. This violates non-triviality. Therefore, $\lambda_0 = 1$.

Solving for λ gives

$$\mathcal{C}u + \mathcal{C}\mathcal{C}^T\lambda = 0 \Rightarrow \lambda = -(\mathcal{C}\mathcal{C}^T)^{-1}\mathcal{C}u$$

$$\Rightarrow \lambda = -(\mathcal{C}\mathcal{C}^T)^{-1}\underbrace{\xi_{NO}}_{\text{from the equality constraint}}$$

Therefore,

$$u = \mathcal{C}^T(\mathcal{C}\mathcal{C}^T)^{-1}\xi_{NO} \quad \text{(from the gradient equation)}$$

By definition of optimality, no other feasible control will have less energy.

Optimization Examples

Example: minimize $(x_1-1)^2 + x_2 - 2$

subj. to $x_1 + x_2 - 2 \leq 0$

$x_2 - x_1 - 1 = 0$

Form the Lagrangian: $L = \lambda_0 (x_1-1)^2 + \lambda_0 x_2 - 2\lambda_0$
$+ \lambda(x_1 + x_2 - 2) + \nu(x_2 - x_1 - 1)$

Compute the gradient of the Lagrangian & set it to zero.

$$\frac{\partial L}{\partial x_1} = 2\lambda_0(x_1-1) + \lambda - \nu = 0 \qquad (1)$$

$$\frac{\partial L}{\partial x_2} = \lambda_0 + \lambda + \nu = 0 \qquad (2)$$

The complementarity condition is $\lambda(x_1 + x_2 - 2) = 0$.
Suppose that $\lambda_0 = 0$.

Eq 1 $\Rightarrow$ $\lambda = \nu$
Eq. 2 $\Rightarrow$ $\lambda = -\nu$

Thus, $\lambda = \nu = 0$. This violates the non-triviality condition.
Therefore $\lambda_0 = 1$.

Suppose $\lambda = 0$.

$$\text{Eq. 2} \Rightarrow \nu = -1$$
$$\text{Eq. 1} \Rightarrow 2(x_1 - 1) = -1 \Rightarrow x_1 = 1/2$$

The equality constraint gives $x_2 = 3/2$. Substituting into the inequality constraint gives $1/2 + 3/2 - 2 = 0 \leq 0$. Therefore, $(x_1, x_2) = (1/2, 3/2)$ is a candidate.

Suppose that $x_1 + x_2 - 2 = 0$. Together with the equality constraint, we have a system of equations

$$x_1 + x_2 = 2$$
$$x_2 - x_1 = 1$$

The solution to this system is again $(x_1, x_2) = (1/2, 3/2)$. Thus, we have only one candidate. If a solution exists, this is it!

Example: minimize $x_1^2 + 4x_2^2$

subj. to $x_1^2 + 2x_2^2 \geq 4$

Form the Lagrangian: $L = \lambda_0 x_1^2 + 4\lambda_0 x_2^2 + \lambda(-x_1^2 - 2x_2^2 + 4)$

Compute the gradient of the Lagrangian:

$$\frac{\partial L}{\partial x_1} = 2\lambda_0 x_1 - 2\lambda x_1 = 0$$

$$\frac{\partial L}{\partial x_2} = 8\lambda_0 x_2 - 4\lambda x_2 = 0$$

The complementarity condition is $\lambda(-x_1^2 - 2x_2^2 + 4) = 0$.
Suppose $\lambda_0 = 0$. Then $\lambda \neq 0$, or else it would violate non-triviality. Therefore, $x_1 = x_2 = 0$. But this does not satisfy the inequality constraint. Therefore, $\lambda_0 = 1$.

Suppose $\lambda = 0$. Then again, $x_1 = x_2 = 0$ which can't be.
Therefore, $x_1^2 + 2x_2^2 = 4$.

$$\text{Eq. 1} \Rightarrow 2x_1(1-\lambda) = 0$$
$$\text{Eq. 2} \Rightarrow 4x_2(2-\lambda) = 0$$

If $\lambda = 1$, then $x_2 = 0$ and $x_1 = \pm 2$. $\rightarrow f = 4$
If $\lambda = 2$, then $x_1 = 0$ and $x_2 = \pm\sqrt{2}$. $\rightarrow f = 8$.
Our only candidate is the $(x_1, x_2) = (\pm 2, 0)$.

Example: minimize $x_2 - (x_1-2)^3 + 3$
subj. to $x_2 \geq 1$

By inspection, what is the answer?

Form the Lagrangian: $L = \lambda_0 x_2 - \lambda_0 (x_1-2)^3 + 3\lambda_0 + \lambda(1-x_2)$

Compute the gradient:

$$\frac{\partial L}{\partial x_1} = -3\lambda_0 (x_1-2)^2 = 0 \qquad (1)$$

$$\frac{\partial L}{\partial x_2} = \lambda_0 - \lambda = 0 \qquad (2)$$

It is evident from Eq. 2 that $\lambda_0 = \lambda = 1$ (else the non-triviality condition would be violated).
Eq. 1 then gives $x_1 = 2$.

The complementarity condition $\lambda(1-x_2) = 0$ gives $x_2 = 1$.

Thus, our only candidate is $(x_1, x_2) = (2,1)$ giving an objective value of 4.

What if I choose $x_1 = 0$? $x_1 = -10$?, $x_1 = -1000$?
The problem does not have a minimum since it is unbounded in x_1.

Example: minimize $(x_1-2)^2 + (x_2-1)^2$

s.t. $x_2 - x_1^2 \geq 0$

$2 - x_1 - x_2 \geq 0$

$x_1 \geq 0.$

The Lagrangian is
$$L = \lambda_0 (x_1-2)^2 + \lambda_0 (x_2-1)^2 + \lambda_1 (x_1^2 - x_2) + \lambda_2 (x_1 + x_2 - 2) + \lambda_3 (-x_1)$$

Compute the gradient of the Lagrangian

$$\frac{\partial L}{\partial x_1} = 2\lambda_0 (x_1-2) + 2\lambda_1 x_1 + \lambda_2 - \lambda_3 = 0.$$

$$\frac{\partial L}{\partial x_2} = 2\lambda_0 (x_2-1) - \lambda_1 + \lambda_2 = 0.$$

Suppose the first two constraints are active. Then

$$\begin{array}{l} x_1^2 = x_2 \\ x_1 + x_2 = 2 \end{array} \Rightarrow \begin{array}{l} x_1 = 1 \\ x_2 = 1 \end{array} \quad \text{or} \quad \cancel{x_1 = -2}$$

Since $x_1 = 1 > 0$, $\lambda_3 = 0$.

$$\text{Eq. 1} \Rightarrow 2\lambda_0(-1) + 2\lambda_1 + \lambda_2 = 0$$
$$\text{Eq. 2} \Rightarrow \lambda_1 = \lambda_2$$

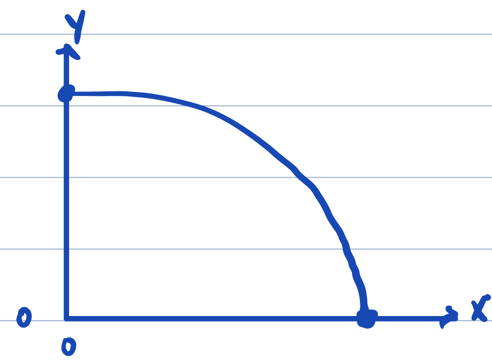
Thus,

$$-2\lambda_0 + 3\lambda_1 = 0$$

If $\lambda_0 = 0$, then $\lambda_1 = \lambda_2 = \lambda_3 = 0$, which violates non-triviality. Therefore $\lambda_0 = 1$, $\lambda_1 = \lambda_2 = 2/3$, $\lambda_3 = 0$. We've found a candidate.

Polynomial Landing Example

We want to design a landing trajectory that looks like the following:



The initial conditions are fixed. The final altitude and velocities are also fixed. The final range & flight time are free. Thus, we have 7 boundary conditions.

We will assume the position trajectories are cubic in time.

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$x = b_0 + b_1 t + b_2 t^2 + b_3 t^3$$

We can differentiate to get the velocity & acceleration profiles.

$$\dot{y} = a_1 + 2a_2 t + 3a_3 t^2$$
$$\dot{x} = b_1 + 2b_2 t + 3b_3 t^2$$
$$\ddot{y} = 2a_2 + 6a_3 t$$
$$\ddot{x} = 2b_2 + 6b_3 t$$

With these, we can now write down the constraints arising from boundary conditions.

(1) $a_0 = y_0$

(2) $b_0 = x_0 = 0$

(3) $a_1 = \dot{y}_0 = 0$

(4) $b_1 = \dot{x}_0$

(5) $a_0 + a_1 T + a_2 T^2 + a_3 T^3 = 0$ $\quad (y(T) = 0)$

(6) $b_1 + 2b_2 T + 3b_3 T^2 = 0$ $\quad (\dot{x}(T) = 0)$

(7) $a_1 + 2a_2 T + 3a_3 T^2 = -1$ $\quad (\dot{y}(T) = -1)$

There are 9 variables and 7 constraints; hence two degrees of freedom. We want to minimize the net acceleration.

$$\begin{aligned} \|F\|^2 &= (2a_2 + 6a_3 t)^2 + (2b_2 + 6b_3 t)^2 \\ &= 4(a_2^2 + b_2^2) + 24(a_2 a_3 + b_2 b_3)t + 36(a_3^2 + b_3^2)t^2 \\ &= c_0 + c_1 t + c_2 t^2 \end{aligned}$$

The integral is then

$$\int_0^T \|F\|\, dt = c_0 T + \tfrac{1}{2} c_1 T^2 + \tfrac{1}{3} c_2 T^3$$

The resulting optimization problem is:

$$\min_{a,b,T} \quad c_0 T + \tfrac{1}{2} c_1 T^2 + \tfrac{1}{3} c_2 T^3$$

$$\text{s.t.} \quad \text{Eqs (1) - (7).}$$

This is a nonlinear programming problem with 9 variables.

Does a global minima exist? Is the Weierstrass Thm. Satisfied?
How can we know if we're finding the global minima?

Well, we've already reduced the problem to a function of two variables. If we fix the flight time T and the final range x_F, then we can solve a sequence of linear equations & generate a contour plot.

(8) $\quad b_0 + b_1 T + b_2 T^2 + b_3 T^3 = x_F$

(9) $\quad T =$ some fixed number

When we do this, we see that the problem with range & flight time free is actually ill-posed in the sense that the cost can be made arbitrarily close to zero by letting $x_F, T \to \infty$, but zero cost cannot be attained.

A similar analysis shows that fixing the range and letting the flight time be free does not correct the issue. The objective has $\inf J = 0$ as $T \to \infty$. (However, such solutions result in unrealistic trajectories.)

One approach that does work is to fix the flight time and optimize the range x_F.

Why are we running into these issues? We did not model the mass dynamics. Hence, the vehicle can fly forever w/o running out of fuel.

While the use of splines (polynomials) is an easy way to generate trajectories, it must be done with caution.

- The NLP solutions are sensitive to the initial guess.
- Global minima do not always exist.
- Resulting trajectories are not always realistic.

Try using splines to solve a LEO transfer problem.

Let's show that $J \to 0$ as $T, x_F \to \infty$. To do so, we will artificially fix T & x_F. And then take the limit.

$$a_0 + a_1 T + a_2 T^2 + a_3 T^3 = 0$$
$$b_1 + 2b_2 T + 3b_3 T^2 = 0$$
$$a_1 + 2a_2 T + 3a_3 T^2 = -1$$
$$b_0 + b_1 T + b_2 T^2 + b_3 T^3 = x_F$$

The values of $a_0, a_1, b_0,$ & b_1 are fixed by the initial conditions. The unknowns are $a_2, a_3, b_2,$ & b_3.

$$\begin{pmatrix} T^2 & T^3 \\ 2T & 3T^2 \end{pmatrix} \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_0 - a_1 T \\ -1 - a_1 \end{pmatrix}$$

$$\Rightarrow \quad a_2 = \left(T - 3a_0 - 2\cancelto{0}{Ta_1}\right) / T^2$$
$$a_3 = -\left(T - 2a_0 - \cancelto{0}{Ta_1}\right) / T^3$$

$$\begin{pmatrix} T^2 & T^3 \\ 2T & 3T^2 \end{pmatrix} \begin{pmatrix} b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} x_F - b_0 - b_1 T \\ -b_1 \end{pmatrix}$$

$$\Rightarrow \quad b_2 = -\left(\cancelto{0}{3b_0} - 3x_F + 2b_1 T\right) / T^2$$
$$b_3 = \left(\cancelto{0}{2b_0} - 2x_F + T b_1\right) / T^3$$

Note that

$$c_0 \sim a_2^2 + b_2^2 \sim 1/T^2$$
$$c_1 \sim a_2 a_3 + b_2 b_3 \sim 1/T^3$$
$$c_2 \sim a_3^2 + b_3^2 \sim 1/T^4$$

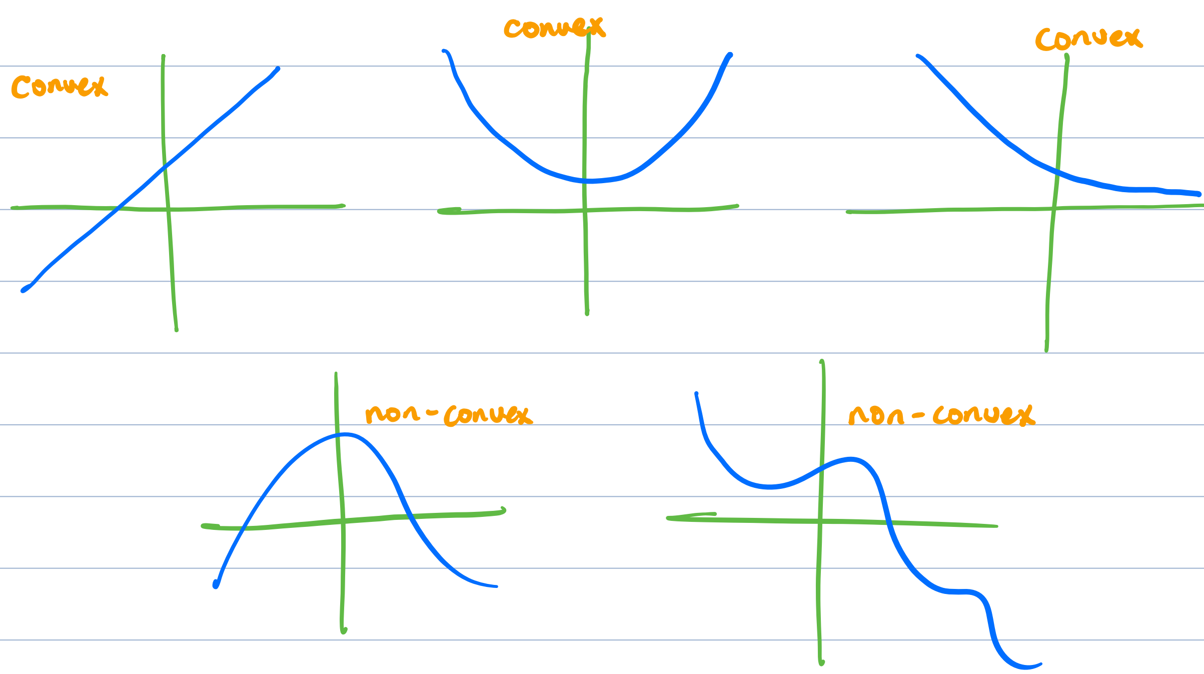
Thus,

$$\lim_{T\to\infty} c_0 T + c_1 T^2 + c_2 T^3 \longrightarrow 0$$

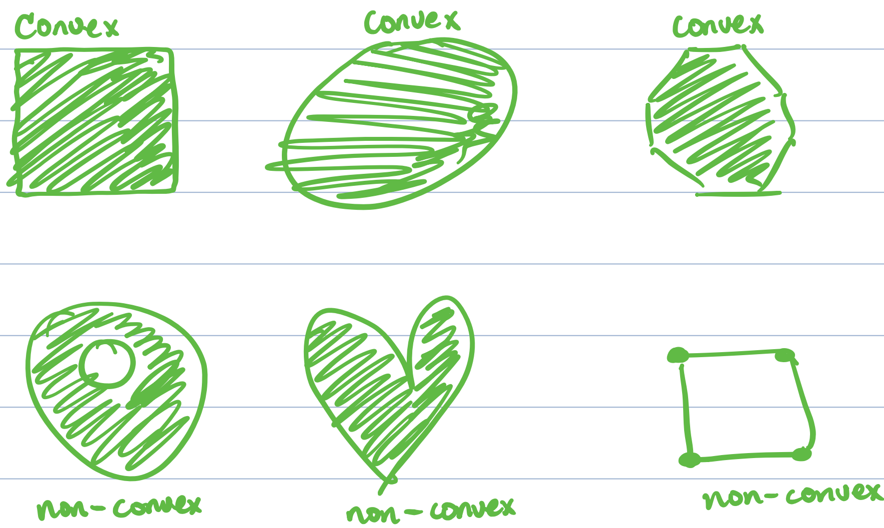
Introduction to Convexity

In many of our previous examples we found candidates but were unsure if they were actually minima. It turns out that convex optimization problems have a rich enough structure to obviate such issues. In some sense, convex problems are the easy ones: both theorems and algorithms are stronger.

Conceptually, convex functions are linear or shaped like bowls.



Conceptually, convex sets are sets without holes or indentations.



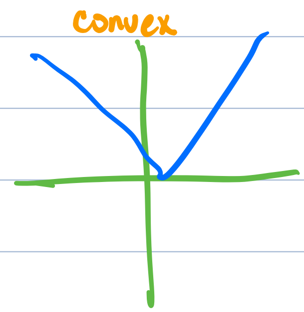
A convex optimization problem is one with a convex objective function and convex constraint functions.

Like linear programs, any optimal solution of a convex program is the globally optimal solution.

Also like linear programs, there may be no solutions, one solution, or multiple solutions.

$$\min \; e^{-x} \quad \text{s.t.} \quad 0 \le x \le \infty, \qquad \min \; x^2 \quad \text{s.t.} \quad -1 \le x \le 1, \qquad \min \; 0x \quad \text{s.t.} \quad -1 \le x \le 1$$

Convex functions do not have to be smooth. The absolute value function is convex.



Definition: A function $f(x)$ is convex if

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

for all x, y and all $\alpha, \beta \geq 0$ + $\alpha + \beta = 1$. That is, the line connecting any two points is not below the curve between those two points. $\square$

See that the blue line is above the green curve between the two points.

Observation: A linear function is convex since

$$f(\alpha x+\beta y) = \alpha f(x) + \beta f(y)$$

for all x, y & α, β.

Example: Is the function $f(x) = x_1 x_2$ convex?

No. Take $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $y = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Then

$$\alpha x + (1-\alpha) y = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2-\alpha \\ 1+\alpha \end{pmatrix}$$

Evaluating the function at this point gives

$$f\big(\alpha x + (1-\alpha) y\big) = (2-\alpha)(1+\alpha) = 2 + \alpha - \alpha^2$$

Evaluating the linear approximation gives

$$\alpha f(x) + (1-\alpha) f(y) = \alpha 2 + (1-\alpha) 2 = 2$$

Is $2 + \alpha - \alpha^2 \leq 2$ for all $\alpha \in (0,1)$? No.
Let $\alpha = 1/2$. Then $2 + 1/2 - 1/4 = 9/4 > 2$.

Example: Show that $f(x) = x^2$ is convex using the definition of a convex function.

$$f(\alpha x + (1-\alpha)y) = (\alpha x + (1-\alpha)y)^2$$
$$= \alpha^2 x^2 + 2\alpha(1-\alpha)xy + (1-\alpha)^2 y^2$$

$$\alpha f(x) + (1-\alpha) f(y) = \alpha x^2 + (1-\alpha) y^2$$

Now, applying the inequality in the definition gives

$$\alpha x^2 + (1-\alpha)y^2 - \alpha^2 x^2 - 2\alpha(1-\alpha)xy - (1-\alpha)^2 y^2 \geq 0$$

$$\Rightarrow \alpha x^2 + y^2 - \alpha y^2 - \alpha^2 x^2 - 2\alpha(1-\alpha)xy - (1 - 2\alpha + \alpha^2) y^2 \geq 0$$
$$\Rightarrow \alpha(1-\alpha)x^2 - 2\alpha(1-\alpha)xy + \alpha(1-\alpha)y^2 \geq 0$$
$$\Rightarrow \alpha(1-\alpha)(x-y)^2 \geq 0$$

The last inequality is true for all x & y and any $\alpha \in [0,1]$.
$\therefore$ $f(x) = x^2$ is convex.

Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}$ & $f \in C^2$. The function f is convex if & only if for every $x \in \mathbb{R}^n$, the Hessian $\nabla^2 f(x)$ is positive semi-definite. $\square$

Proof: Expand f using a Taylor series & apply the definition of convexity. $\square$

Example: Determine if $f(x) = x^2$ is convex.

$$\nabla f = 2x$$
$$\nabla^2 f = 2 \geq 0 \Rightarrow \text{convexity}$$

Example: Determine if $f(x) = x^3$ on $[0, \infty)$ is convex.

$$\nabla f = 3x^2$$
$$\nabla^2 f = 6x \geq 0 \text{ on } [0,\infty) \Rightarrow \text{convexity.}$$

Example: Determine if $f(x) = x_1^2 - x_2^2$ is convex.

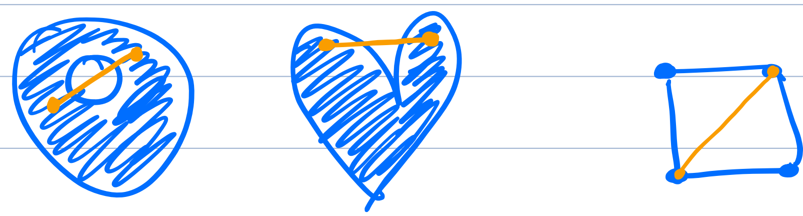
$$\nabla f = [2x_1, -2x_2]$$
$$\nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

The $\text{eig}(\nabla^2 f) = \pm 2$. Thus, $\nabla^2 f \not\succeq 0$. $\Rightarrow$ non-convexity.

Definition: A set C is convex if the line segment between any two points in C lies in C, i.e., if for any $x_1, x_2 \in C$ and any $\theta \in [0,1]$, we have

$$\theta x_1 + (1-\theta)x_2 \in C.$$

Graphical Examples:



non-convex non-convex non-convex

These are non-convex because the orange line segment connects two points in C but goes outside of C.

Example: Show that the line segment $S = \{x : 0 \preceq x \preceq 1\}$ is convex. Let x_1 and x_2 be any two points in S. The point $\theta x_1 + (1-\theta)x_2$ is in the closed interval between x_1 and x_2, which is contained in S. Therefore, $\theta x_1 + (1-\theta)x_2 \in S$, and the set is convex.

Example: Show that the boundary of the unit cube is non-convex. Let $x_1 = (0,0)$ and $x_2 = (1,1)$. Let $\theta = 1/2$. Then $\theta x_1 + (1-\theta)x_2 = (1/2, 1/2)$, which is not in the set.

Definition: The convex hull of a set S, denoted $\text{conv}\, S$, is the set of all convex combinations of points in S.

$$\text{conv}\, S = \left\{ \theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k : x_i \in S,\ \theta_i \geq 0,\ \sum_{i=1}^{k} \theta_i = 1 \right\} \quad \square$$

Graphical Examples: Green denotes S. Blue denotes $\text{conv}\, S$.

The convex hull of S is a "convex relaxation" of S.
It is also the "tightest" convex relaxation.

Other Properties:

- Intersection: If S_1 + S_2 are convex, then $S_1 \cap S_2$ is convex.

- Separation: Suppose S_1 + S_2 are two convex sets + $S_1 \cap S_2 = \phi$.
  Then there is an $a \neq 0$ and b s.t.

$$a^T x \leq b \quad \forall \; x \in S_1$$

and $$a^T x \geq b \quad \forall \; x \in S_2$$

In other words, the two sets can be separated by a hyperplane.

- Non-negative Sum: Suppose $w_i \geq 0$ + f_i is convex for all i
  Then $\sum w_i f_i$ is convex.

- Affine mapping: Suppose f is convex. Then $f(Ax+b) = g(x)$ is convex.

Conditions for Convex Programming

We are interested in developing necessary conditions for convex optimization problems. We start w/ a general problem. We sometimes call this the primal problem.

$$\min_x \quad f(x)$$
$$\text{s.t.} \quad g(x) \leq 0$$
$$h(x) = 0 \qquad (Ax+b=0 \text{ for convex problems})$$

To help in this endeavor, we will define "the Lagrangian"

$$L(x,\lambda,\nu) = f(x) + \lambda^T g(x) + \nu^T h(x)$$

The new variables λ & ν are called dual variables or Lagrange multipliers. There is a dual variable for every constraint in the problem.

We now define the dual function ℓ.

$$\ell(\lambda,\nu) = \inf_x L(x,\lambda,\nu) = \inf_x \Big(f(x) + \lambda^T g(x) + \nu^T (Ax-b) \Big)$$

The dual function is always concave — even when the primal problem is non-convex.

Why do we care about the dual function? It provides a lower bound for our optimization problem. Let p^* denote the optimal objective value for our problem.

Lemma: For any $\lambda \geq 0$ and any ν, $\ell(\lambda,\nu) \leq p^*$. $\square$

Proof: To see this, let $\hat{x}$ be a feasible point, i.e.,

$$g(\hat{x}) \leq 0 \quad \text{and} \quad h(\hat{x}) = A\hat{x} - b = 0.$$

Then, it is easy to see that

$$\underbrace{\underset{\geq 0}{\lambda^T} \underset{\leq 0}{g(\hat{x})}}_{\leq 0} + \nu^T \underbrace{h(\hat{x})}_{=0} \leq 0$$

Therefore, $L(\hat{x},\lambda,\nu) = f(\hat{x}) + \lambda^T g(\hat{x}) + \nu^T h(\hat{x}) \leq f(\hat{x})$, which is true for any feasible $\hat{x}$ and $\lambda \geq 0$.
By taking the infimum w.r.t x, the value of the Lagrangian can only be decreased. Thus,

$$\ell(\lambda,\nu) = \inf_x L(x,\lambda,\nu) \leq L(\hat{x},\lambda,\nu) \leq f(\hat{x})$$

Since $\ell(\lambda,\nu) \leq f(\hat{x})$ for all feasible $\hat{x}$, it follows that $\ell(\lambda,\nu) \leq p^* = f(x^*)$. $\square$

The fact that we just proved is called "weak duality". The dual function provides a lower bound for our problem.

When is this lower bound tight - meaning when is the dual function equal to our objective value? To answer this question, we need to maximize our dual function.

This is called the dual problem.

$$\max_{\lambda,\nu} \; \ell(\lambda,\nu) \quad \text{s.t.} \quad \lambda \geq 0.$$

We'll call the optimal objective value for this dual problem d^*.

To summarize so far: For every (primal) optimization problem, there is a dual problem s.t. $d^* \leq p^*$.

The positive quantity $p^* - d^*$ is called the <u>duality gap</u>.

<u>Strong duality</u> is said to hold when $p^* = d^*$.

With an assumption that strong duality holds, we can state a very generic set of optimality conditions known as the Karush-Kuhn-Tucker (KKT) conditions.

Theorem: Assume that f, g, h are differentiable.

If 1) the optimization problem attains a minimum at x^*,

2) the dual attains a maximum at (λ^*, v^*),

3) strong duality holds,

Then the following system is solvable:

(1) $g(x^*) \leq 0$

(2) $h(x^*) = 0$

(3) $\lambda^* \geq 0$

(4) $\lambda^{*T} g(x^*) = 0$

(5) $\nabla_x f(x^*) + \nabla_x g(x^*) \lambda^* + \nabla_x h(x^*) v^* = 0$ $\square$

Under the three assumptions stated, these are the necessary conditions for optimality of any optimization problem!

Let's see why the theorem is true.

Proof: Assumption 1 tells us that $g(x^*) \leq 0$ + $h(x^*) = 0$.
Assumption 2 tells us that $\lambda^* \geq 0$.
Assumption 3 tells us that

$$
\begin{aligned}
f(x^*) &= \ell(\lambda^*, \nu^*) \\
&= \inf_x \left(f(x) + \lambda^{*T} g(x) + \nu^{*T} h(x) \right) && \text{by def'n of the dual function.} \\
&\leq f(x^*) + \lambda^{*T} g(x^*) + \nu^{*T} h(x^*) && \text{because inf minimizes a function} \\
&\leq f(x^*) && \text{since } h = 0 \text{ + } \lambda^T g \leq 0.
\end{aligned}
$$

The 1st + 4th lines obviously hold w/ equality.
Thus, the 3rd line does, too. We can now deduce 2 facts:

1. The point x^* also minimizes $L(x, \lambda^*, \nu^*)$.
2. The product $\lambda^{*T} g(x^*) = 0$.

Since the problem of minimizing $L(x, \lambda^*, \nu^*)$ is unconstrained,

$$\nabla_x L(x, \lambda^*, \nu^*) = \nabla_x f(x^*) + \nabla_x g(x^*) \lambda^* + \nabla_x h(x^*) \nu^* = 0.$$

This concludes the proof! $\square$

Assumptions 2) & 3) are somewhat odd since they don't directly involve our optimization problem. They involve the dual problem, which was a mathematical construction.

How can we ever verify assumptions 2 & 3?

There is an entire sub-area of optimization devoted to this. It is called "constraint qualifications."

For linear & convex problems, we can easily state when assumptions 2 & 3 hold.

Lemma: Suppose the optimization problem is linear. If the problem is feasible, then assumptions 2 & 3 hold. (i.e., the dual attains a maximum & strong duality holds.) □

Proof: See the PDF online. □

For convex problems, we need to know what a strictly feasible point is.

Definition: A point $\hat{x}$ is strictly feasible if $g(\hat{x}) < 0$. $h(\hat{x}) = 0$ □

Theorem (Slater's Constraint Qualification): Suppose that the optimization problem is convex & has finite objective value. If there is a strictly feasible point, then assumptions 2) & 3) hold. (i.e., the dual attains a maximum & strong duality holds.) □

Proof: See the PDF online.

To summarize so far, we have necessary conditions for optimality called the KKT conditions. They involve 3 assumptions. For linear & convex problems, we have "nice" constraint qualifications to tell us when assumptions 2 & 3 are satisfied.

You may recall that the optimality conditions for linear programs were necessary & sufficient. We'll now prove this for convex programs.

Theorem (KKT conditions for Convex Programming):
Suppose that f + g are differentiable. If Slater's CQ holds, the optimization problem attains a minimum at x <u>if + only if</u> the following system is solvable:

$$g(x) \leq 0$$
$$Ax = b$$
$$\lambda \geq 0$$
$$\lambda^T g(x) = 0$$
$$\nabla_x f(x) + \nabla_x g(x)\lambda + A^T v = 0 \qquad \square$$

Proof: We've already proved the "necessity." We'll now prove "sufficiency." Suppose that the system is solvable. The first two equations imply that x is feasible.

Because the Lagrangian is convex in x and its derivative w.r.t. x is zero at x, it attains a minimum there. Thus,

$$\begin{aligned} \ell(\lambda, v) &= L(x, \lambda, v) \\ &= f(x) + \lambda^T g(x) + v^T(Ax - b) \\ &= f(x) \end{aligned}$$

This shows that the duality gap is zero. So x must be minimizing f. $\square$

We've now proved the necessary & sufficient optimality conditions for convex problems (& hence linear problems)!

Example: minimize $\frac{1}{2}x^T P x + q^T x + r$, $P = P^T \geq 0$
subj. to $Ax = b$

Note that Slater's CQ is trivially satisfied since there are no inequality constraints. The Lagrangian is

$$L(x,\lambda,v) = \frac{1}{2}x^T P x + q^T x + r + v^T(Ax - b)$$

The gradient of the Lagrangian is

$$\nabla_x L = Px + q + A^T v = 0$$

Therefore, the solution to this problem is obtained by solving the linear system

$$\begin{pmatrix} P & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} -q \\ b \end{pmatrix}$$

Any solution of this matrix equation will be a global minimizer.

This leads to Newton's Method w/ linear equality constraints. We are interested in minimizing a nonlinear function

$$\min \; f(x) \quad \text{s.t.} \quad Ax = b.$$

As we did before for Newton's Method, we'll take a 2^{nd}-order Taylor approximation.

$$\min_{v} \; f(x+v) = f(x) + \nabla_x^T f(x)\, v + \frac{1}{2} v^T \nabla^2 f(x)\, v$$

$$\text{s.t.} \quad A(x+v) = b$$

We want to choose v to minimize our quadratic approximation & satisfy the equality constraint.

If x (our current point in Newton's method) is feasible,

$$Ax = b, \quad \text{then} \quad Av = 0.$$

Using the results from the previous example, v must satisfy

$$\begin{pmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} v \\ \upsilon \end{pmatrix} = \begin{pmatrix} -\nabla f(x) \\ 0 \end{pmatrix} \qquad (\Delta)$$

Algorithm: Given a feasible starting point x & tolerance $\varepsilon \geq 0$

repeat:

1) Compute the Newton step v from Eq. Δ and Newton decrement $\lambda(x) = \left(v^T \nabla^2 f(x) v\right)^{1/2}$
2) Stop if $\lambda^2/2 \leq \varepsilon$
3) Line Search for t.
4) Update $x = x + tv$.

This algorithm is described in Ch. 10.2 (p. 525–528) of the book by Boyd.

Example of the dual function:

Primal Problem : $\min \ x^T x \quad \text{s.t.} \quad Ax = b$

Write down the Lagrangian

$$L=(x,v) = x^T x + v^T(Ax-b)$$

The dual function is given by $g(v) = \inf_x L(x,v)$. Since L is a quadratic convex function of x, we can find the minimizing x by taking the gradient to zero.

$$\nabla_x L = 2x + A^T v = 0 \quad \Rightarrow \quad x = -\tfrac{1}{2} A^T v$$

Substituting this back into L gives

$$\begin{aligned} g(v) &= \tfrac{1}{4} v^T A A^T v - \tfrac{1}{2} v^T A A^T v - v^T b \\ &= -\tfrac{1}{4} v^T A A^T v - b^T v \end{aligned}$$

This is a concave function in v. Weak duality states that

$$-\tfrac{1}{4} v^T A A^T v - b^T v \leq \inf\{x^T x \mid Ax = b\} \quad \text{for any } v.$$

Discrete Optimal Control

We are now interested in solving an optimization problem whose constraints include a discrete dynamic system.

$$\text{minimize} \quad J = \phi(x_N) + \sum_{k=0}^{N-1} \ell^k(x_k, u_k)$$

$$\text{subj. to} \quad x_{k+1} = f^k(x_k, u_k), \quad k = 0, \dots, N-1$$

$$x_0 \text{ is specified}, \quad \psi(x_N) = 0$$

The function $\phi(x_N)$ is the "terminal cost." It is a penalty on the states only at the final time. For example, if we want to drive a system close to the origin,

$$\phi(x_N) = \| x_N \|.$$

The function $\ell^k(x_k, u_k)$ is the "running cost." It penalizes states & controls all along the trajectory (except at the final time). For example, if we want to keep the states & controls close to zero, then

$$\ell(x_k, u_k) = \tfrac{1}{2} x_k^T x_k + \tfrac{1}{2} u_k^T u_k.$$

The dynamics of the system are $x_{k+1} = f^k(x_k, u_k)$. The initial condition of the system is fixed at x_0, and the final state of the system is not fixed, but constrained by $\psi(x_N) = 0$.

At present, we are not constraining the controls, i.e., $u_k \in \mathbb{R}^m$.

We can apply the Fritz John conditions to arrive at optimality conditions specific to this problem. (We'll use λ^0 to denote our abnormal multiplier.)

The Lagrangian is

$$L = \lambda^0 \phi(x_N) + \lambda^0 \sum_{k=0}^{N-1} \ell^k(x_k, u_k) + \sum_{k=0}^{N-1} \lambda_{k+1}^T \left(f^k(x_k, u_k) - x_{k+1} \right)$$

$$+ \nu^T \psi(x_N)$$

(see that there are N λ's: $\lambda_1, \ldots, \lambda_N$)

Before we move on to computing partials, it will be convenient to define the Hamiltonian

$$H^k(x_k, u_k, \lambda^0, \lambda_{k+1}) = \lambda^0 \ell^k(x_k, u_k) + \lambda_{k+1}^T f^k(x_k, u_k).$$

for $k = 0, \ldots, N-1$.

The Lagrangian is then

$$L = \lambda^0 \phi(x_N) + \nu^T \psi(x_N) + \sum_{k=0}^{N-1} H^k(x_k, u_k, \lambda^0, \lambda_{k+1}) - \sum_{k=1}^{N} \lambda_k^T x_k$$

$$= \lambda^0 \phi(x_N) + \nu^T \psi(x_N) + H^0(x_0, u_0, \lambda^0, \lambda_1) - \lambda_N^T x_N$$

$$+ \sum_{k=1}^{N-1} \left(H^k(x_k, u_k, \lambda^0, \lambda_{k+1}) - \lambda_k^T x_k \right)$$

We now compute partial derivatives and set them equal to zero.

$$\frac{\partial L}{\partial x_N} = \lambda^0 \frac{\partial \phi}{\partial x_N} + \frac{\partial \psi}{\partial x_N} \nu - \lambda_N = 0$$ (Transversality Condition)

$$\frac{\partial L}{\partial u_k} = \frac{\partial H^k}{\partial u_k} = 0$$ (Stationarity Condition) $k = 0, \dots, N-1$

$$\frac{\partial L}{\partial x_k} = \frac{\partial H^k}{\partial x_k} - \lambda_k = 0$$ (costate equation) $k = 1, \dots, N-1$

$$x_{k+1} = \frac{\partial H^k}{\partial \lambda_{k+1}} = f^k(x_k, u_k)$$ (state equation) $k = 0, \dots, N-1$

How do these conditions change when there is a control constraint of the form $u_k \in \Omega^k = \{ w \in \mathbb{R}^m : q^k(w) \preceq 0 \}$?

Example: Let's look at the scalar minimum control energy problem with linear dynamics.

$$\text{minimize} \quad J = \frac{1}{2}\sum_{k=0}^{N-1} u_k^2$$

$$\text{subj. to} \quad x_{k+1} = a x_k + b u_k \;, \quad x_0 \text{ is given}$$
$$x_N \text{ is given}$$

Recall, the goal is to find the control sequence $u_0, u_1, \ldots, u_{N-1}$ to drive the state from x_0 to x_N.

Begin by writing the Hamiltonian and the optimality conditions.

$$H^k = \frac{1}{2}\lambda^0 u_k^2 + \lambda_{k+1}(a x_k + b u_k)$$
$$\lambda_k = a\lambda_{k+1} \qquad \text{(costate equation)}$$
$$0 = u_k + b\lambda_{k+1} \qquad \text{(stationarity condition)}$$

Note that the transversality condition is trivially satisfied in this problem.

Solving for u_k in the stationarity condition gives

$$u_k = -b\lambda_{k+1}. \qquad (\text{assuming } \lambda_0 = 1)$$

If we can find λ_{k+1}, we can find the optimal control. In an attempt to do so, eliminate u_k in the state equation resulting in

$$x_{k+1} = a x_k - b^2 \lambda_{k+1}$$

Now, recognize that the costate equation is a simple recursion with solution $\lambda_k = a^{N-k} \lambda_N$ s.t.

$$x_{k+1} = a x_k - b^2 a^{N-k-1} \lambda_N$$

We've now reduced the problem to finding λ_N.

The solution to the above equation is

$$x_k = a^k x_0 - b^2 a^{N+k-2} \lambda_N \sum_{j=0}^{k-1} a^{-2j}$$

The summation is a geometric series whose sum can be written explicitly. We'll take another approach later.

$$x_k = a^k x_0 - b^2 a^{N+k-2} \lambda_N \frac{(1 - a^{-2k})}{(1 - a^{-2})}$$

$$= a^k x_0 - b^2 a^{N-k} \lambda_N \frac{(1 - a^{2k})}{(1 - a^2)}$$

At the final time, we get

$$x_N = a^N x_0 - b^2 \lambda_N \frac{(1 - a^{2N})}{(1 - a^2)}$$

$$= a^N x_0 - \Lambda \lambda_N \quad \left[\text{where } \Lambda = b^2 \frac{(1 - a^{2N})}{(1 - a^2)}\right]$$

Solving for λ_N gives

$$\lambda_N = \frac{1}{\Lambda}\left(a^N x_0 - x_N\right).$$

Substituting this back into the costate equation gives

$$\lambda_k = \frac{1}{\Lambda}\left(a^N x_0 - x_N\right) a^{N-k}$$

At last, the optimal control is

$$\boxed{u_k = -b \lambda_{k+1} = \frac{-b}{\Lambda}\left(a^N x_0 - x_N\right) a^{N-k-1}}$$

This control will drive the (a,b) system from x_0 to x_N in N steps & minimize the control energy required to do so!

To obtain the above expression we used the formula for a geometric series. If we start back at

$$x_k = a^k x_0 - b^2 a^{N+k-2} \lambda_N \sum_{j=0}^{k-1} a^{-2j}$$

and replace k with N, then

$$x_N = a^N x_0 - b^2 a^{2N-2} \lambda_N \sum_{j=0}^{N-1} a^{-2j}$$

By redefining $\Lambda = b^2 a^{2N-2} \sum_{j=0}^{N-1} a^{-2j}$, we can write

$$x_N = a^N x_0 - \Lambda \lambda_N.$$

Solving for λ_N gives

$$\lambda_N = \frac{1}{\Lambda}\left(a^N x_0 - x_N\right).$$

And from here, the analysis is the same.

Why did we use the geometric formula in the first place? Only because it provides a clean formula for Λ that is easier to implement.

Note that when $|a| > 1$, the numerical values explode quickly leading to numerical issues.

Connecting Optimization & Discrete Optimal Control

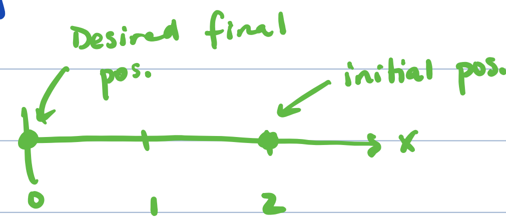
We started the course by studying Fritz John & KKT conditions for optimization problems.

We then moved to discrete optimal control. We derived optimality conditions for such a problem (in terms of a Hamiltonian) by using the Fritz John conditions. I hope it is evident that the two sets of conditions are equivalent. They are just written in different forms.

Let's imagine a 1-D problem. The goal is to move an object from its starting position $x_0 = 2$ to a final position of $x_2 = 0$ in 2 steps. This means that at time 0 the object can be moved and at time 1 the object can be moved. After this second move, the object should be at zero.

The amounts that we move are denoted by u_0 & u_1. Thus, the object moves according to

$$x_1 = x_0 + u_0$$
$$x_2 = x_1 + u_1$$



We can combine the equations as

$$x_2 = x_0 + u_0 + u_1$$

Since x_2 and x_0 are known, we group them together

$$x_2 - x_0 = -2 = u_0 + u_1$$

Any movements u_0 & u_1 that add to -2 are feasible movements.

Of all the feasible movements, let's find the ones requiring the least amount of energy given by

$$J = \frac{1}{2}(u_0^2 + u_1^2)$$

We'll solve the problem 3 ways:

The Easiest Way: Solve the problem

$$\min_{u_0, u_1} \frac{1}{2}(u_0^2 + u_1^2)$$

$$\text{s.t.} \quad u_0 + u_1 + 2 = 0.$$

The Lagrangian is

$$L = \frac{\lambda_0}{2}(u_0^2 + u_1^2) + \nu(u_0 + u_1 + 2)$$

Compute the partials

$$\frac{\partial L}{\partial u_0} = \lambda_0 u_0 + \nu = 0$$

$$\frac{\partial L}{\partial u_1} = \lambda_0 u_1 + \nu = 0$$

If $\lambda_0 = 0$, then $\nu = 0$ violating non-triviality.
Thus, $\lambda_0 = 1$ and $u_0 = u_1 = -\nu$.

Since $u_0 + u_1 = -2$ and $u_0 = u_1$, we conclude that

$$u_0 = u_1 = -1$$

Using the FJ Conditions: We now solve the problem with the x's still included.

$$\min_{u_0, u_1, x_1} \quad \frac{1}{2}(u_0^2 + u_1^2)$$
$$\text{s.t.} \quad x_1 = x_0 + u_0$$
$$x_2 = x_1 + u_1$$

The Lagrangian is

$$L = \frac{\lambda_0}{2}(u_0^2 + u_1^2) + \lambda_1(x_0 + u_0 - x_1) + \lambda_2(x_1 + u_1 - x_2)$$

The partial derivatives are

$$\frac{\partial L}{\partial u_0} = \lambda_0 u_0 + \lambda_1 = 0$$

$$\frac{\partial L}{\partial u_1} = \lambda_0 u_1 + \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_1} = -\lambda_1 + \lambda_2 = 0$$

If $\lambda_0 = 0$, then $\lambda_1 = \lambda_2 = 0$. Thus, $\lambda_0 = 1$. We again see that

$$u_0 = u_1 \quad \text{and} \quad u_0 + u_1 = -2$$

Thus, $\quad u_0 = u_1 = -1$

Using the Hamiltonian Conditions

We now use the discrete optimal control conditions.
Write the Hamiltonian

$$H^k = \frac{\lambda_0}{2} u_k^2 + \lambda_{k+1} f^k(x_k, u_k)$$

$$\Downarrow$$

$$H^0 = \frac{\lambda_0}{2} u_0^2 + \lambda_1 (x_0 + u_0)$$

$$H^1 = \frac{\lambda_0}{2} u_1^2 + \lambda_2 (x_1 + u_1)$$

The stationarity conditions are

$$\frac{\partial H^0}{\partial u_0} = \lambda_0 u_0 + \lambda_1 = 0$$

$$\frac{\partial H^1}{\partial u_1} = \lambda_0 u_1 + \lambda_2 = 0$$

Note these are the same conditions we had on the previous page!

The costate equations are

$$\lambda_1 = \frac{\partial H^1}{\partial x_1} = \lambda_2$$

Note it's the same again!

Since the conditions are the same: $u_0 = u_1 = -1$

Discrete Linear Quadratic Control

Let's now investigate the matrix-vector LQR problem.

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}\sum_{k=0}^{N-1} u_k^T R\, u_k \,, \qquad R > 0 \\ &\text{s.t.} \qquad x_{k+1} = A x_k + B u_k \,, \quad x_0 \ \&\ x_N \text{ given} \end{aligned}$$

As before, begin by writing the Hamiltonian & optimality conditions.

$$H^k = \frac{1}{2}\lambda^0 u_k^T R\, u_k + \lambda_{k+1}^T (A x_k + B u_k)$$

$$\lambda_k = A^T \lambda_{k+1}$$

$$0 = \lambda^0 R u_k + B^T \lambda_{k+1}$$

Solving for the control in the stationarity condition gives

$$u_k = -R^{-1} B^T \lambda_{k+1} \qquad (\text{assuming } \lambda^0 = 1)$$

What would happen in the abnormal case where $\lambda^0 = 0$?

We can then substitute this into the state dynamics to get

$$x_{k+1} = A x_k - B R^{-1} B^T \lambda_{k+1}$$

One can show (i.e., you should show) that

$$\lambda_k = A^{T(N-k)} \lambda_N$$

such that the state equation can be written in terms of λ_N.

$$x_{k+1} = A x_k - B R^{-1} B^T A^{T(N-k-1)} \lambda_N$$

Writing out a few terms in the sequence gives...

$$x_1 = A x_0 - B R^{-1} B^T A^{T(N-1)} \lambda_N$$

$$\begin{aligned} x_2 &= A x_1 - B R^{-1} B^T A^{T(N-2)} \lambda_N \\ &= A^2 x_0 - A B R^{-1} B^T A^{T(N-1)} \lambda_N - B R^{-1} B^T A^{T(N-2)} \lambda_N \end{aligned}$$

From here, we can deduce the following:

$$x_k = A^k x_0 - \sum_{i=0}^{k-1} A^{k-i-1} B R^{-1} B^T A^{T(N-i-1)} \lambda_N$$

We now set $k=N$ to find an expression for x_N.

$$x_N = A^N x_0 - \sum_{i=0}^{N-1} A^{N-i-1} B R^{-1} B^T A^{T(N-i-1)} \lambda_N$$

Defining the summation to be Λ gives

$$x_N = A^N x_0 - \Lambda \lambda_N$$

Solving for λ_N (provided Λ^{-1} exists) gives

$$\lambda_N = \Lambda^{-1}(A^N x_0 - x_N)$$

Thus, $\lambda_k = A^{T(N-k)} \Lambda^{-1}(A^N x_0 - x_N)$, and the optimal control is

$$\begin{aligned} u_k &= -R^{-1} B^T \lambda_{k+1} \\ &= -R^{-1} B^T A^{T(N-k-1)} \Lambda^{-1} (A^N x_0 - x_N) \end{aligned}$$

This control will drive the (A, B) system from x_0 to x_N in N steps & minimize the control energy required.

As an exercise:

- Discretize the CWH equations.
- Pick x_0, x_N, and N.
- Implement the optimal control to verify it works.
- Solve the problem numerically using Yalmip & see if you get the same answer.

The above solution requires that Λ^{-1} exists. What is the meaning of this requirement? To see, let's write out Λ.

$$\begin{aligned}\Lambda = \; & A^{N-1} B R^{-1} B^T A^{T(N-1)} \\ + \; & A^{N-2} B R^{-1} B^T A^{T(N-2)} \\ + \; & \cdots \\ + \; & A^0 B R^{-1} B^T A^{T(0)}\end{aligned}$$

$$= \left[B, AB, \dots, A^{N-1}B\right] \begin{bmatrix} R^{-1} & & 0 \\ & \ddots & \\ 0 & & R^{-1} \end{bmatrix} \left[B, AB, \dots, A^{N-1}B\right]^T$$

The matrix $C = \left[B, AB, \dots, A^{N-1}B\right]$ is the controllability matrix! Thus, if we drive the (A,B) system from x_0 to x_N with the above control law if $\text{rank}(C) = n$, i.e., if the system is controllable.

What is a weaker condition than having Λ^{-1} exist?

Let's now investigate the problem with terminal objective.

$$\text{minimize} \quad \tfrac{1}{2} x_N^T S_N x_N + \tfrac{1}{2} \sum_{k=0}^{N-1} u_k^T R u_k \,, \quad R \succ 0,\ S \succeq 0$$

$$\text{s.t.} \quad x_{k+1} = A x_k + B u_k$$

$$x_0 \text{ is given (but not } x_N)$$

As before, begin by writing the Hamiltonian & optimality conditions.

$$H^k = \tfrac{1}{2} \lambda^0 u_k^T R u_k + \lambda_{k+1}^T (A x_k + B u_k)$$

$$\lambda_k = A^T \lambda_{k+1} \,, \quad \lambda_N = S_N x_N$$

Transversality Condition

$$0 = \lambda^0 R u_k + B^T \lambda_{k+1}$$

Solving for the control in the stationarity condition gives

$$u_k = -R^{-1} B^T \lambda_{k+1} \quad (\text{assuming } \lambda^0 = 1)$$

We can then substitute this into the state dynamics to get

$$x_{k+1} = A x_k - B R^{-1} B^T \lambda_{k+1}$$

Following the same logic, we arrive at

$$x_N = A^N x_0 - \Lambda \lambda_N$$

We can now impose the transversality condition $\lambda_N = S_N x_N$ to get

$$x_N = A^N x_0 - \Lambda S_N x_N$$

$$\Rightarrow \quad (I + \Lambda S_N) x_N = A^N x_0$$

Provided the inverse exists, we can solve for the final state.

$$x_N = (I + \Lambda S_N)^{-1} A^N x_0$$

Observe that when $S_N = 0$, the final state becomes unimportant, and $x_N = A^N x_0$, which arises from unforced motion. Thus, $u_k = 0 \ \forall k$.

Suppose $S_N = \gamma I$. What happens to x_N as $\gamma \to \infty$?

If we denote u_{Free} as the optimal control from the free final state problem and u_{Fix} as the optimal control from the fixed final state problem, what can we say about the relative magnitudes of

$$\sum u_{Fix}^T R\, u_{Fix} \quad \text{and} \quad \sum u_{Free}^T R\, u_{Free} \;?$$

Discrete LQ Regulator

We now consider the discrete LQR problem with a running cost on the state. We also let the final state be free, but penalize it as well.

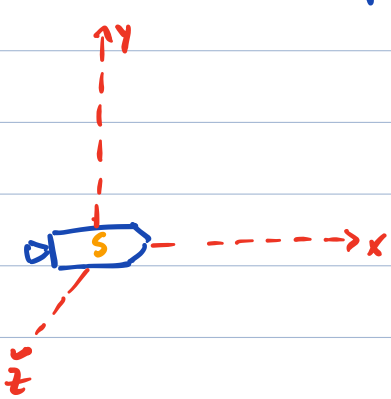
$$\min \quad \frac{1}{2} x_N^T S_N x_N + \frac{1}{2} \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k .$$

$$\text{subj. to} \quad x_{k+1} = A x_k + B u_k$$

We require that $S_N = S_N^T \geq 0$, $Q = Q^T \geq 0$, $R = R^T > 0$.

Also, the $A, B, Q,$ & R could be time-varying, but we don't do that here for simplicity.

As motivation, consider a spacecraft trying to "regulate" its state – or drive it close to 0.



If the spacecraft S is at the origin and the origin is a stationary point, no control is needed. Otherwise some control is needed to drive it to the origin & keep it there.

To analyze this problem, we write the Hamiltonian.

$$H^k = \tfrac{1}{2}\lambda_0\left(x_k^T Q x_k + u_k^T R u_k\right) + \lambda_{k+1}^T\left(Ax_k + Bu_k\right)$$

we'll assume $\lambda_0 = 1$ again.

The costate dynamics and transversality conditions are

$$\lambda_k = Qx_k + A^T\lambda_{k+1}, \quad \lambda_N = S_N x_N$$

The stationarity condition is

$$\frac{\partial H^k}{\partial u_k} = Ru_k + B^T\lambda_{k+1} = 0 \Rightarrow u_k = -R^{-1}B^T\lambda_{k+1}$$

The techniques we used before no longer work since the recursion for λ_k is no longer homogenous. A method introduced by Bryson & Ho is the sweep method.

Since $\lambda_N = S_N x_N$, assume there are matrices S_k s.t.

$$\lambda_k = S_k x_k \quad \forall \; k \le N.$$

Now, we need to find formulas for S_k.

Substituting into the state equation gives

$$x_{k+1} = A x_k - B R^{-1} B^T S_{k+1} x_{k+1}$$

Solving for x_{k+1} gives

$$x_{k+1} = \left(I + B R^{-1} B^T S_{k+1}\right)^{-1} A x_k ,$$

which is a forward, homogenous recursion for the state.
Substituting $\lambda_k = S_k x_k$ into the costate equation gives

$$\begin{aligned} S_k x_k &= Q x_k + A^T S_{k+1} x_{k+1} \\ &= Q x_k + A^T S_{k+1} \left(I + B R^{-1} B^T S_{k+1}\right)^{-1} A x_k \end{aligned}$$

Since this must hold for all x_k, we see that

$$S_k = Q + A^T S_{k+1} \left(I + B R^{-1} B^T S_{k+1}\right)^{-1} A$$

Another way to write this (using the matrix inversion lemma) is

$$S_k = Q + A^T \left(S_{k+1} - S_{k+1} B \left(B^T S_{k+1} B + R \right)^{-1} B^T S_{k+1} \right) A$$

The above equation is known as the Riccati equation.

Since we know S_N, we can find all S_k. We can then write the control

$$u_k = -R^{-1} B^T S_{k+1} x_{k+1}.$$

We are almost there, but u_k depends on x_{k+1}, which is a future state.

$$u_k = -R^{-1} B^T S_{k+1} \left(A x_k + B u_k \right)$$

$$\Rightarrow \left(I + R^{-1} B^T S_{k+1} B \right) u_k = -R^{-1} B^T S_{k+1} A x_k$$

Pre-multiplying by R and inverting gives

$$u_k = -\left(R + B^T S_{k+1} B \right)^{-1} B^T S_{k+1} A x_k$$

We now define the Kalman gain as

$$K_k = \left(R + B^T S_{k+1} B \right)^{-1} B^T S_{k+1} A$$

so that the control is simply $u_k = -K_k x_k$. Note that the Kalman gain is time-varying even though $A, B, Q, \&\ R$ are time-invariant. This is a feedback control law since it depends on our current state x_k — not the initial state x_0.

While simple to implement, we still have to store a sequence of S matrices.

Is it possible to come up with a single S (and hence constant feedback matrix)?

One approach is to consider very long time horizons where $N-k \to \infty$. If the S_k recursion reaches steady state, then $S_k = S_{k+1} \equiv S$. The above Riccati equation becomes the Algebraic Riccati Equation (ARE).

$$S = Q + A^T\left(S - SB\left(B^TSB + R\right)^{-1}B^TS\right)A$$

The Kalman gain is then constant.

① When does the limit exist?
② When is S independent of S_N?
③ When is the closed-loop system stable?

Informal Theorem: The above hold when

(A, C) is observable where $Q = C^TC$

and (A, B) is stabilizable. $\square$

How can we find the steady state matrix S?

- One approach is to pick an S_N and iterate backward until a steady state is reached.

- MATLAB has a built-in command "idare".

How can this be used to track nonlinear dynamics such as a spacecraft in orbit?

The dynamics of a nonlinear system are given by

$$\dot{x} = f(x,u)$$

We want this system to follow some pre-computed (optimal) trajectory denoted by x^*, u^*. Linearize about this:

$$\delta\dot{x} = \nabla_x f(x^*,u^*)\,\delta x + \nabla_u f(x^*,u^*)\,\delta u$$

Then discretize (e.g. using Euler integration)

$$\delta x_{k+1} = \delta x_k + h\left(\nabla_x f(x_k^*,u_k^*)\,\delta x_k + \nabla_u f(x_k^*,u_k^*)\,\delta u_k\right)$$

$$= \underbrace{\left[I + h\nabla_x f(x_k^*,u_k^*)\right]}_{A_k}\delta x_k + \underbrace{h\nabla_u f(x_k^*,u_k^*)}_{B_k}\,\delta u_k$$

$$= A_k\,\delta x_k + B_k\,\delta u_k$$

Thus, the "perturbed" dynamics are linear + time-varying.
We can solve the LQR problem for δx_k + δu_k.
The actual control we apply is then $u_k = u_k^* + \delta u_k$.

Discrete LQ Tracking

In the previous notes, we developed a feedback controller to "regulate" the dynamics, i.e., keep the state close to zero. We will now develop a feedback controller to "track" a reference output trajectory.

Reference trajectory r_k is one that may not depend on all states such that our goal is to use little control and have $Cx_k \cong r_k$.

The discrete optimal control problem that models this problem is:

$$\min \quad \frac{1}{2}(Cx_N - r_N)^T P (Cx_N - r_N)$$

$$+ \frac{1}{2}\sum_{k=0}^{N-1}\left[(Cx_k - r_k)^T Q (Cx_k - r_k) + u_k^T R u_k\right]$$

$$\text{subj. to} \quad x_{k+1} = Ax_k + Bu_k$$

To begin analyzing the problem, write the Hamiltonian.

$$H^k = \frac{\lambda^0}{2}(Cx_k - r_k)^T Q (Cx_k - r_k) + \frac{\lambda^0}{2} u_k^T R u_k$$

$$+ \lambda_{k+1}(Ax_k + Bu_k)$$

We'll assume that $\lambda^0 = 1$. The costate equation is

$$\lambda_k = A^T \lambda_{k+1} + C^T Q C x_k - C^T Q r_k$$

The stationarity condition is

$$0 = R u_k + B^T \lambda_{k+1} \quad \Rightarrow \quad u_k = -R^{-1} B^T \lambda_{k+1}$$

The transversality condition is

$$\lambda_N = C^T P (C x_N - r_N) = \underbrace{C^T P C}_{S_N} x_N - \underbrace{C^T P r_N}_{v_N}$$

As we did before, we'll use a sweep method whereby we assume

$$\lambda_k = S_k x_k - v_k$$

The control equation is then

$$u_k = -R^{-1} B^T \left(S_{k+1} x_{k+1} - v_{k+1} \right)$$

$$\Rightarrow \quad x_{k+1} = A x_k - B R^{-1} B^T S_{k+1} x_{k+1} + B R^{-1} B^T v_{k+1}$$

Solving for x_{k+1} gives

$$x_{k+1} = (I + BR^{-1}B^T S_{k+1})^{-1}(Ax_k + BR^{-1}B^T v_{k+1})$$

Using this in the costate equation gives

$$\underline{S_k x_k} - \underline{v_k} = C^T Q C x_k - C^T Q r_k + A^T(S_{k+1}x_{k+1} - v_{k+1})$$

$$= \underline{C^T Q C x_k} - \underline{C^T Q r_k} - \underline{A^T v_{k+1}}$$

$$+ \underline{A^T S_{k+1}(I + BR^{-1}B^T S_{k+1})^{-1}(Ax_k} + BR^{-1}B^T \underline{v_{k+1}})$$

Grouping all of the x_k terms and non-x_k terms gives

$$\left[-S_k + C^T Q C + A^T S_{k+1}(I + BR^{-1}B^T S_{k+1})^{-1}A\right]x_k$$

$$+\left[v_k - C^T Q r_k - A^T v_{k+1} + A^T S_{k+1}(I + BR^{-1}B^T S_{k+1})^{-1}BR^{-1}B^T v_{k+1}\right] = 0$$

Since this must hold for all x_k, both terms need to be zero. The first term lets us find S_k as a function of S_{k+1}. The second term lets us find v_k as a function of S_{k+1} and v_{k+1}.

The optimal control is then

$$u_k = -R^{-1} B^T \lambda_{k+1}$$

$$= -R^{-1} B^T \left(S_{k+1} x_{k+1} - v_{k+1} \right)$$

$$= -R^{-1} B^T S_{k+1} \left(A x_k + B u_k \right) + R^{-1} B^T v_{k+1}$$

Pre-multiply by R and solve for u_k.

$$u_k = \left(R + B^T S_{k+1} B \right)^{-1} B^T \left(-S_{k+1} A x_k + v_{k+1} \right)$$

We can make things look nicer if we define the

Feedback Gain: $K_k = \left(R + B^T S_{k+1} B \right)^{-1} B^T S_{k+1} A$

Feedforward Gain: $K_k^v = \left(R + B^T S_{k+1} B \right)^{-1} B^T$

s.t. $u_k = -K_k x_k + K_k^v v_{k+1}$.

How would things change if the system were time varying, i.e., we had A_k and B_k?

When the dynamics are time-invariant we can look for sub-optimal constant feedback gains. As before, if (A,B) is reachable and $(A, C\sqrt{Q})$ is observable, then the recursions for K_k and K_k^v reach steady state at $N-k \to \infty$.

The constant gains are then

$$K = (B^T S_\infty B + R)^{-1} B^T S_\infty A$$

$$K^v = (B^T S_\infty B + R)^{-1} B^T$$

$$u_k = -K x_k + K^v v_{k+1}$$

It appears that we have to still store the v_k sequence. But we don't. Instead, store v_0 and then propagate forward using

$$v_{k+1} = (A - BK)^{-T} v_k - (A-BK)^{-T} C^T Q r_k .$$

Discretization of Nonlinear Systems

By assuming piecewise constant controls, the discretization and numerical solution of optimization problems with linear dynamics is relatively straightforward.

Given a nonlinear function $f : D \rightarrow \mathbb{R}$, it may be possible to decompose it into linear combinations of basis functions $T_i(t)$:

$$f(t) = \sum_{i=0}^{\infty} a_i T_i(t)$$

You are probably already familiar with this concept from linear algebra. For example,

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\hat{\imath}} + 3 \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{\jmath}}$$

basis vectors

Just as there are many bases for $\mathbb{R}^2$, there are many bases for a function space. In these notes, we will use the Chebyshev polynomials as the basis functions.

Chebyshev Polynomials

These polynomials are defined on the domain $[-1, 1]$ and given by the formula

$$T_0(t) = 1$$
$$T_1(t) = t$$
$$T_{n+1}(t) = 2t\,T_n(t) - T_{n-1}(t)$$

Example: $T_2(t) = 2t(t) - 1 = 2t^2 - 1$

$$T_3(t) = 2t(2t^2-1) - t$$
$$= 4t^3 - 2t - t$$
$$= 4t^3 - 3t$$

We also have Chebyshev polynomials of the second kind

$$U_0(t) = 1$$
$$U_1(t) = 2t$$
$$U_{n+1}(t) = 2t\,U_n(t) - U_{n-1}(t)$$

Both kinds of Chebyshev polynomials form an orthogonal basis.

The two kinds of polynomials are related by

$$2T_n(t) = U_n(t) - U_{n-2}(t), \quad T_n(t) = U_n(t) - t\,U_{n-1}(t)$$

They satisfy a number of interesting properties.

$$T_n(1) = 1$$
$$T_n(-1) = (-1)^n$$
$$U_n(1) = n+1$$
$$U_n(-1) = (-1)^n (n+1)$$

Their derivatives are also related.

$$\dot{T}_i(t) = i\,U_{i-1}(t)$$

$$\dot{U}_i(t) = \frac{(i+1)\,T_{i+1}(t) - t\,U_n(t)}{t^2-1}$$

There are other properties, too.

Suppose we discretize the interval $[-1, 1]$ into $n+1$ nodes $t_0, t_1, \ldots, t_{n+1}$. The function values are $f(t_i)$.
We can then use the first $n+1$ polynomials to approximate the function.

$$f(t) \cong \sum_{i=0}^{n} a_i T_i(t)$$

However, we must first solve for the a_i values. This is easily done.

$$\underbrace{\begin{bmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_n) \end{bmatrix}}_{\bar{f}} = \underbrace{\begin{bmatrix} T_0(t_0) & T_1(t_0) & \cdots \\ T_0(t_1) & T_1(t_1) & \cdots \\ \vdots & \vdots & \\ T_0(t_n) & T_1(t_n) & \cdots \end{bmatrix}}_{\Upsilon} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}}_{\bar{a}}$$

$$\Rightarrow \quad \bar{a} = \Upsilon^{-1} \bar{f}$$

After computing each $\dot{T}_i(t_j)$, we can also approximate the derivative of f.

$$\dot{f}(t) \cong \sum_{i=0}^{n} a_i \dot{T}_i(t)$$

Using the same matrix notation as above, evaluation at the nodes gives

$$\dot{\bar{f}} = \dot{\mathcal{T}}\bar{a}$$

$$= \underbrace{\dot{\mathcal{T}}\mathcal{T}^{-1}}_{D}\bar{f}$$

$$= D\bar{f}$$

That is, there is a matrix D that maps the function values at nodes t_i to derivative values at the nodes. This matrix is called the "Differentiation Matrix."

If we have a choice in the node selection process, we can choose them in a way to minimize the approximation error. The optimally placed nodes are called the "Chebyshev nodes."

$$x_k = \cos\left(\frac{\pi}{2}\frac{(2k-1)}{n+1}\right), \quad k = 1, \ldots, n+1$$

It is common for these polynomials to be used in optimization and boundary value problems. However, the nodes do not always land at the end points.

Therefore, they are some times approximated as

$$x_j = \cos\left(\pi \frac{j}{n}\right), \quad j = 0, \dots, n$$

Note that the differentiation matrix depends on the node selection.

Example: Using the approximate Chebyshev nodes, compute $\mathcal{T}$, $\dot{\mathcal{T}}$, and D with $n = 2$.

The above formula tells us that $t_0 = -1$, $t_1 = 0$, $t_2 = +1$. The $\mathcal{T}$ matrix is given by

$$\mathcal{T} = \begin{bmatrix} T_0(t_0) & T_1(t_0) & T_2(t_0) \\ T_0(t_1) & T_1(t_1) & T_2(t_1) \\ T_0(t_2) & T_1(t_2) & T_2(t_2) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Its derivative is given by

$$\dot{\mathcal{T}} = \begin{bmatrix} \dot{T}_0(t_0) & \dot{T}_1(t_0) & \dot{T}_2(t_0) \\ \dot{T}_0(t_1) & \dot{T}_1(t_1) & \dot{T}_2(t_1) \\ \dot{T}_0(t_2) & \dot{T}_1(t_2) & \dot{T}_2(t_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

The differentiation matrix is

$$D = \dot{T}\, T^{-1} = \frac{1}{2}\begin{bmatrix} -3 & 4 & -1 \\ -1 & 0 & 1 \\ 1 & -4 & 3 \end{bmatrix}$$

Introduction to Optimal Control

We will optimize continuous systems whose evolution is governed by ordinary differential equations. If you recall, our foray into discrete optimal control was motivated by such systems. In continuous optimal control, we no longer need to discretize to analyze the problem.

A standard optimal control problem is:

$$\min \quad J = \phi(t_f, x_f) + \int_{t_0}^{t_f} L(t, x, u)\, dt$$

$$\text{subj. to} \quad \dot{x} = f(t, x, u), \quad x_0 \text{ is given}$$

$$\psi(t_f, x_f) = 0, \quad u(t) \in \Omega$$

As in discrete optimal control:

- ϕ is the terminal or Mayer cost
- L is the running or Lagrange cost
- f is the system dynamics w/ initial condition x_0
- ψ is the terminal constraint
- Ω is the control constraint

Example: The goal is to achieve a minimum time rendezvous using the CWH Equations and bounded control.

$$\min \; t_f \qquad \text{or} \qquad \min \int_{t_0}^{t_f} 1 \, dt$$

$$\text{subj. to} \quad \dot{x} = Ax + Bu, \quad x_0 \text{ given}$$
$$\psi(t_f, x_f) = x_f = 0$$
$$\|u(t)\| \leq \rho$$

Example: The minimum fuel rendezvous has objective

$$\min \int_{t_0}^{t_f} \|u(t)\| \, dt.$$

Example: The minimum energy rendezvous has objective

$$\min \int_{t_0}^{t_f} \|u(t)\|^2 \, dt.$$

Example: The LQR objective is

$$\min \; \tfrac{1}{2} x^T(t_f) S_f \, x(t_f) + \tfrac{1}{2} \int_{t_0}^{t_f} x^T(t) Q \, x(t) + u^T(t) R \, u(t) \, dt$$

The necessary conditions for an optimal control problem are proved in another set of notes. They are commonly called the maximum principle, Pontryagin's Principle, ...

The conditions are: If x and u are minimizers, then there exists a $\underbrace{(\lambda_0, \lambda(t)) \neq 0}_{\text{non-triviality}}$ with $\underbrace{\lambda_0}_{\text{abnormal multiplier}} \in \{0,1\}$ such that

$$H = \lambda_0 L + \lambda^T f$$ Hamiltonian

$$G = \lambda_0 \phi + \nu^T \psi$$ Endpoint Function

$$\dot{x} = \frac{\partial H}{\partial \lambda} = f$$ State Equation

$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$ Costate Equation

$$\left.\begin{aligned} \lambda(t_f) &= \frac{\partial G}{\partial x_f} \\ H(t_f) &= -\frac{\partial G}{\partial t_f} \end{aligned}\right\}$$ Transversality Conditions

$$u \in \underset{w \in \Omega}{\arg\min}\ H(t, x, w)$$ Pointwise minimum condition.

A comment on vector derivatives: Given a scalar-valued vector function $f : \mathbb{R}^n \to \mathbb{R}$ we denote its gradient as

$$\frac{\partial f}{\partial x} = \nabla_x f = \partial_x f = f_x = \begin{bmatrix} \partial f / \partial x_1 \\ \vdots \\ \partial f / \partial x_n \end{bmatrix}$$ which is $n \times 1$.

Given a vector-valued vector function $\Psi : \mathbb{R}^n \to \mathbb{R}^m$, we denote its gradient as

$$\frac{\partial \Psi}{\partial x} = \nabla_x \Psi = \partial_x \Psi = \Psi_x = \begin{bmatrix} \partial \Psi_1 / \partial x_1 & \cdots & \partial \Psi_m / \partial x_1 \\ \vdots & \cdots & \vdots \\ \partial \Psi_1 / \partial x_n & & \partial \Psi_m / \partial x_n \end{bmatrix}$$

which is $n \times m$.

So let's expand the following:

costate equation: $\dot{\lambda} = -\frac{\partial H}{\partial x} = -\lambda_0 \frac{\partial L}{\partial x} - \frac{\partial f}{\partial x} \lambda$

transversality conditions: $\lambda(t_f) = \frac{\partial G}{\partial x_f} = \lambda_0 \frac{\partial \phi}{\partial x_f} + \frac{\partial \Psi}{\partial x_f} \nu$

$$H(t_f) = -\frac{\partial G}{\partial t_f} = -\lambda_0 \frac{\partial \phi}{\partial t_f} - \frac{\partial \Psi}{\partial t_f} \nu$$

Many authors use other conventions. This one is the "cleanest."

Example: Let's think about a simple car on a straight track trying to reach the finish line as quickly as possible.

$$\min \; t_f$$
$$\text{s.t.} \quad \dot{x} = u, \quad x_0 = 0, \quad x_f = 1, \quad -1 \le u \le 1$$

The solution procedure is to form the Hamiltonian & endpoint functions.

$$H = \lambda u$$
$$G = \lambda_0 t_f + \nu (x_f - 1)$$

Then start going through the conditions.

$$\dot{x} = u$$
$$\dot{\lambda} = 0 \qquad \text{(means } \lambda \text{ is constant)}$$
$$\lambda(t_f) = \nu \qquad \text{(gives no useful info)}$$
$$H(t_f) = -\lambda_0 = \lambda(t_f)\, u(t_f)$$
$$u(t) = \underset{-1 \le w \le 1}{\arg\min} \; \lambda w \quad \Rightarrow \quad u(t) = \begin{cases} -1 & \lambda > 0 \\ +1 \;, & \lambda < 0 \\ \text{singular} & \lambda = 0 \end{cases}$$

If $\lambda = 0$, then $\lambda_0 = 0$ violating non-triviality. Thus, "singular" solutions cannot occur. The optimal control is either always -1 or always $+1$.

If $u=-1$, then $x=-t$. Since $t \geq 0$, we can't satisfy the final condition $x(t_f)=1$.

If $u=+1$, then $x=t$. The final condition is satisfied when $t_f=1$.

$$\boxed{\text{The optimal control is } u(t)=1 \;\; \forall\, t \in [0,1].}$$

Example: Let's minimize the energy to move the car from $x_0=0$ to $x_f=1$. Ignore the control constraint.

$$\min \int_0^{t_f} \tfrac{1}{2} u^2 \, dt$$

$$\text{s.t.} \quad \dot{x} = u, \quad x_0 = 0, \quad x_f = 1$$

We first form the Hamiltonian & endpoint functions:

$$H = \frac{\lambda_0}{2} u^2 + \lambda u$$

$$G = \nu (x_f - 1)$$

The optimality conditions are

$\dot{x} = u$

$\dot{\lambda} = 0$

$\lambda_f = \nu$

$H_f = 0 = \frac{\lambda_0}{2} u_f^2 + \lambda_f u_f$

$u(t) = \underset{\omega}{\arg\min} \; \frac{\lambda_0}{2}\omega^2 + \lambda\omega$

Suppose that $\lambda_0 = 0$. The $H_f = 0$ condition implies $u_f = 0$ (since λ_f cannot be zero). When $\lambda_0 = 0$, the pointwise minimum condition reduces to $u = -\infty$ or $+\infty$, inconsistent with $u_f = 0$. Thus, $\lambda_0 = 1$. In this case, the quadratic function is minimized when its derivative is zero, i.e., $u = -\lambda$. The $H_f = 0$ condition indicates $\frac{1}{2}u^2 - u^2 = -\frac{1}{2}u^2 = 0 \Rightarrow u = 0$.

But $u = 0$ won't satisfy the boundary conditions!

The infimum is zero but a min does not exist. This is the equivalent of trying to minimize e^{-x}. Although motivated by a real problem, the problem is ill-posed.

Let's now fix the final time at $t_f = 2$. Then

$$\psi(t_f, x_f) = \begin{pmatrix} x_f - 1 \\ t_f - 2 \end{pmatrix} = 0$$

The endpoint function changes to

$$G = \nu_1 (x_f - 1) + \nu_2 (t_f - 2)$$

The transversality condition becomes

$$H(t_f) = \nu_2$$

Everything else remains the same, and we need to find a constant control that goes from 0 to 1 in 2 seconds.

$$\boxed{\text{The optimal control is } u(t) = 1/2 \quad \forall\, t \in [0, 2].}$$

Example: Let's now look at the minimum fuel problem.

$$\min \int_0^2 |u(t)|\, dt$$

$$\text{s.t.} \quad \dot{x} = u, \quad x_0 = 0, \quad x_f = 1, \quad t_f = 2, \quad -1 \le u \le 1$$

The Hamiltonian and endpoint functions are

$$H = \lambda_0 |u| + \lambda u$$

$$G = \nu_1 (x_f - 1) + \nu_2 (t_f - 2)$$

$$\dot{\lambda} = 0$$

$$\left.\begin{aligned} \lambda_f &= \nu_1 \\ H_f &= \nu_2 \end{aligned}\right\}$$ yields no useful info since ν_1 and ν_2 are unknown.

$$u \in \underset{-1 \le w \le 1}{\arg\min}\ \lambda_0 |w| + \lambda w$$

If $\lambda_0 = 0$, then $u = \begin{cases} -1, & \lambda > 0 \\ +1, & \lambda < 0 \\ \text{sing.}, & \lambda = 0 \end{cases}$

Note that this singular case cannot occur since it would violate non-triviality. Also, $u = -1$ and $u = +1$ do not satisfy the boundary conditions. Thus, $\lambda_0 = 1$.

To satisfy the pointwise minimum condition, we need to minimize $|u| + \lambda u$ subj. to $-1 \le u \le 1$.

- If $\lambda > 1$, then $u = -1$. This won't take us to the final point.
- If $-1 < \lambda < 1$, then $u = 0$. This won't take us to the final point either.
- If $\lambda < -1$, then $u = 1$. This won't take us to the final point either.

- If $\lambda = \pm 1$, the minimizing control is non-unique. Another singular case! But this is our only option. It must be that $\lambda = -1$.

Any control $u(t) \in [0,1]$ $\forall$ t $[0,2]$ will be an "extremal" control, i.e., a candidate for an optimal control.

Let's list a few options:

$$u(t) = 1/2 \ \ \forall \ t \quad \longrightarrow \quad J = 1$$

$$u(t) = \begin{cases} 0, & t \in [0,1] \\ 1, & t \in [0,1] \end{cases} \quad \longrightarrow \quad J = 1$$

There are many more solutions. Like regular optimization problems, optimal control problems may have no solutions (see above), one solution (see above), or infinitely many solutions (this problem).

How do I know $J=1$ is actually the optimal? Because the problem is convex. Solve this as a discrete optimal control problem to see this numerically.

Example: Let's look at a scalar minimum control energy problem with linear dynamics.

$$\min \quad \frac{1}{2}\int_0^{t_f} u^2\,dt$$

$$\text{s.t.} \quad \dot{x} = ax + bu \,, \quad x_0, x_f, t_f \text{ given (with bars on top)}$$

The Hamiltonian & endpoint functions are

$$H = \frac{\lambda_0}{2}u^2 + \lambda(ax+bu)$$

$$G = \nu_1(x_f - \bar{x}_f) + \nu_2(t_f - \bar{t}_f)$$

The optimality conditions are

$$\dot{x} = \frac{\partial H}{\partial \lambda} = ax + bu \qquad \dot{\lambda} = -\frac{\partial H}{\partial x} = -a\lambda$$

$$\lambda(t_f) = \frac{\partial H}{\partial x_f} = \nu_1$$

$$H(t_f) = -\frac{\partial H}{\partial t_f} = \nu_2$$

$$u(t) \in \underset{w}{\arg\min} \; \frac{\lambda_0}{2}w^2 + \lambda b w$$

Suppose $\lambda_0 = 0$. The pointwise minimum condition implies $u = \pm\infty$, which is infeasible & would give infinite cost. Thus, $\lambda_0 = 1$.

The quadratic function is minimized when its gradient is zero, i.e., $u = -b\lambda$.

Substituting into the state equation gives

$$\dot{x} = ax - b^2\lambda$$

Since the costate is homogenous, its solution is given by

$$\lambda(t) = e^{a(t_f - t)}\lambda_f$$

making the state equation

$$\dot{x} = ax - b^2 e^{a(t_f - t)}\lambda_f$$

The solution to this equation is

$$x(t) = e^{a(t-0)}x_0 - \int_0^t e^{a(t-\tau)}\, b^2 e^{a(t_f - \tau)}\lambda_f \, dt$$

Evaluating at the final time gives

$$x_f = e^{at_f} x_0 - \int_0^{t_f} e^{a(t_f-\tau)} b^2 e^{a(t_f-\tau)} \lambda_f \, d\tau$$

$$= e^{at_f} x_0 - \Lambda \lambda_f$$

We can now solve for λ_f (provided $\Lambda \neq 0$)

$$\lambda_f = \frac{1}{\Lambda}\left(e^{at_f} x_0 - x_f\right)$$

Substituting this back into the costate equation gives

$$\lambda = \frac{e^{a(t_f-t)}}{\Lambda}\left(e^{at_f} x_0 - x_f\right)$$

The optimal control is then

$$u = -\frac{b}{\Lambda} e^{a(t_f-t)}\left(e^{at_f} x_0 - x_f\right).$$

This control will drive the (a,b) system from x_0 to x_f in t_f time & minimize the control energy required to do so.

Note how similar the process is to the discrete example.

Example: Vehicle p is pursuing vehicle e, which is evading by moving to the right with constant velocity V_e.

The pursuer starts at the origin. The evader starts at the point (ℓ, h). The dynamics of the pursuer are

$$\dot{x} = u \quad , \quad \dot{u} = a\cos\theta \qquad (\text{horizontal})$$
$$\dot{y} = v \quad , \quad \dot{v} = a\sin\theta \qquad (\text{vertical})$$

The known constant thrust magnitude is a. The control variable is the thrust angle θ. The pursuer wants to intercept the evader as quickly as possible.

minimize t_f subj. to above ODEs.

$$y(t_f) = h$$
$$x(t_f) = \ell + V_e t_f$$

Thus, $\phi = t_f$ and $\psi = \begin{pmatrix} x_f - \ell - V_e t_f \\ y_f - h \end{pmatrix} = 0$

The Hamiltonian and endpoint functions are

$$H = \lambda_1 u + \lambda_2 v + \lambda_3 a\cos\theta + \lambda_4 a\sin\theta$$

$$G = \lambda_0 t_f + \nu_1 (x_f - \ell - V_e t_f) + \nu_2 (y_f - h)$$

The optimality conditions are

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{u} = a\cos\theta, \quad \dot{v} = a\sin\theta$$

$$\dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = 0, \quad \dot{\lambda}_3 = -\lambda_1, \quad \dot{\lambda}_4 = -\lambda_2$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\lambda_1 \text{ const}, \quad \lambda_2 \text{ const}, \quad \lambda_3 = -\lambda_1(t_f - t) + \lambda_{3_f}, \quad \lambda_4 = -\lambda_2(t_f - t) + \lambda_{4_f}$$

$$\lambda_{1_f} = \nu_1, \quad \lambda_{2_f} = \nu_2, \quad \lambda_{3_f} = 0, \quad \lambda_{4_f} = 0$$

$$H_f = -\lambda_0 + \nu_1 V_e = \nu_1 u_f + \nu_2 v_f$$

$$\theta(t) \in \underset{\omega}{\arg\min}\ \lambda_3 a\cos\omega + \lambda_4 a\sin\omega$$

$$\Rightarrow \quad \lambda_3 a\sin\theta = \lambda_4 a\cos\theta \quad \Rightarrow \quad \tan\theta = \frac{\lambda_4}{\lambda_3}$$

Using the costate & transversality conditions together, we see that

$$\lambda_3 = -\nu_1 (t_f - t), \quad \lambda_4 = -\nu_2 (t_f - t)$$

Thus, called the bilinear tangent law.

$$\tan\theta = \boxed{\frac{-v_2(t_f - t)}{-v_1(t_f - t)}} = \frac{v_2}{v_1}$$

That is, the optimal thrust angle is constant. We can now easily integrate the state equations.

$$u = at\cos\theta \quad , \quad x = \tfrac{1}{2}at^2\cos\theta$$
$$v = at\sin\theta \quad , \quad y = \tfrac{1}{2}at^2\sin\theta$$

At the final time, we must have

$$\left.\begin{aligned} \tfrac{1}{2}a t_f^2 \cos\theta &= \ell + V_e t_f \\ \tfrac{1}{2}a t_f^2 \sin\theta &= h \end{aligned}\right\} \Rightarrow \tan\theta = \frac{h}{\ell + V_e t_f}$$

The only thing remaining is to find the optimal final time. One way to do this is to square both sides in the above equations and add.

$$\tfrac{1}{4}a^2 t_f^4 \left(\sin^2\theta + \cos^2\theta\right) = h^2 + (\ell + V_e t_f)^2$$

$$\Rightarrow \tfrac{1}{4}a^2 t_f^4 = h^2 + \ell^2 + 2\ell V_e t_f + V_e^2 t_f^2$$

This quartic equation can be solved for t_f – the minimum intercept control.

Non-singular Minimum Time Control

We are now going to investigate optimal control problems beyond the LQR paradigm. LQR problems are important - especially in tracking type problems. Many problems do not fit that structure.

Example: This is the minimum time control of a double integrator. All quantities are scalars.

$$\begin{aligned}
\min \quad & \int_0^{t_f} 1\,dt \\
\text{s.t.} \quad & \dot{x}_1 = x_2 \,, \quad x_1(0) = x_{10} \,, \quad x_1(t_f) = 0 \\
& \dot{x}_2 = u \,, \quad x_2(0) = x_{20} \,, \quad x_2(t_f) = 0 \\
& |u| \leq 1
\end{aligned}$$

We begin the analysis by forming H & G.

$$H = \lambda_0 + \lambda_1 x_2 + \lambda_2 u$$
$$G = \nu_1 (x_{1f} - 0) + \nu_2 (x_{2f} - 0)$$

The costate & transrversality conditions are

$$\dot{\lambda}_1 = \frac{-\partial H}{\partial x_1} = 0 \,, \quad \lambda_{1f} = \frac{\partial G}{\partial x_{1f}} = \nu_1$$

$$\dot{\lambda}_2 = \frac{-\partial H}{\partial x_2} = -\lambda_1 \,, \quad \lambda_{2f} = \frac{\partial G}{\partial x_{2f}} = \nu_2$$

$$H_f = -\frac{\partial G}{\partial t_f} = 0.$$

The pointwise minimum condition gives

$$u \in \underset{|w| \le 1}{\arg\min}\ \lambda_2 w = \begin{cases} -1, & \lambda_2 > 0 \\ +1, & \lambda_2 < 0 \\ \text{singular} & \lambda_2 \equiv 0 \end{cases}$$

Let's first investigate the singular case. Assume that $\lambda_2 = 0$ on some non-trivial interval of time.

$$\lambda_2 = 0 \Rightarrow \dot{\lambda}_2 = 0 \Rightarrow \lambda_1 = 0$$

(as seen from the costate equations.) At the final time,

$$H_f = \lambda_0 + \cancelto{0}{\lambda_1} x_2 + \cancelto{0}{\lambda_2} u = 0 \Rightarrow \lambda_0 = 0.$$

This violates non-triviality. We conclude that the singular case cannot occur.

As a result, the control can take only values of ± 1. Observe that λ_2 is a linear function of time meaning it can only switch signs one time. Denote such a switch time as t_1.

Thus, there are 4 possible control solutions.

$$u = \begin{cases} +1 & \forall\, t \in [t_0, t_f] \\ -1 & \forall\, t \in [t_0, t_f] \\ +1 & \forall\, t \in [t_0, t_1), \quad -1 \quad \forall\, t \in [t_1, t_f] \\ -1 & \forall\, t \in [t_0, t_1), \quad +1 \quad \forall\, t \in [t_1, t_f]. \end{cases}$$

Because it is λ_2 that is causing the control to switch, it is some times called the "switching function."

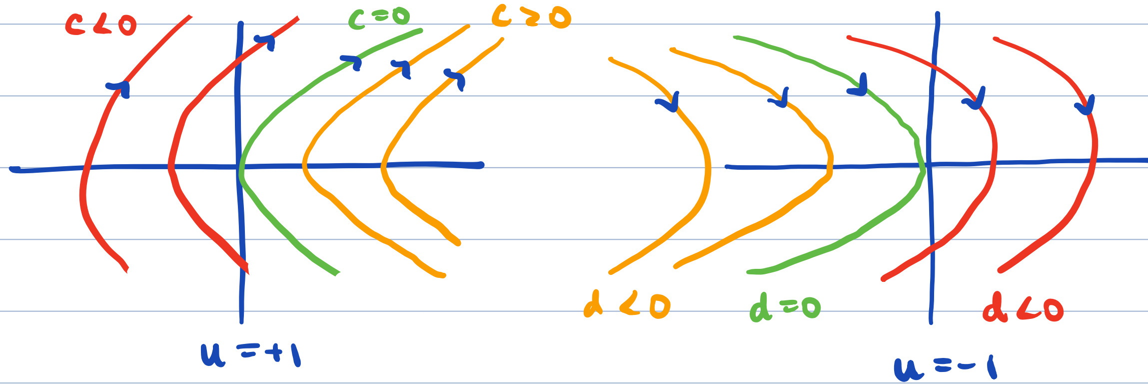
Integrating the state equations with $u = \pm 1$ gives

$$x_2 = \pm t + a$$
$$x_1 = \pm \frac{1}{2} t^2 + at + b$$

where $a = x_2(0)$ and $b_2 = x_2(0)$. Eliminating t gives

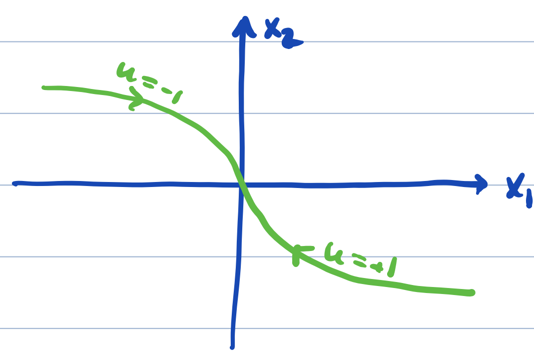
$$x_1 = +\frac{1}{2} x_2^2 + c \qquad \text{for} \quad u = +1$$
$$x_1 = -\frac{1}{2} x_2^2 + d \qquad \text{for} \quad u = -1$$

We can then plot the parabolas for various values of c + d.



Now pause and think about these graphs. If we start in the 1st quadrant, applying $u=+1$ will move us farther from the origin. Applying $u=-1$ will move us into the 4th quadrant. As soon as we hit the green ($u=+1$) curve in the 4th quadrant, we can "switch" to $u=+1$ and go straight to the origin.

This motivates the following switching curve. $x_1 = -\frac{1}{2} x_2 |x_2|$



If the current state is above the switching curve, apply a control of $u=-1$. If the current state is below the switching surface, apply a control of $u=+1$. If the current state is on the switching curve and x_2 is positive (negative), apply $u=-1$ ($u=+1$).

Since we've solved the problem for any current state, this constitutes a feedback control law. This is the best possible situation.

We'll now explore two other ways we could solve this problem. They will result in open-loop solutions meaning the solution is specific to the initial state.

A Sequential Convex Program: For a fixed final time, we could discretize and solve in Yalmip. If Yalmip returns infeasible, we know our final time is too small. If Yalmip returns a feasible answer, then the minimum final time is less than or equal to the final time used.

Thus, we need to solve a sequence of Yalmip problems searching for the least final time for which the problem is feasible.

This type of approach is called a Direct Method. It involves only the states & controls. It does not involve the costates.

The Shooting Method: The shooting method is an Indirect Method. It uses the costates, and it tries to solve the optimality conditions.

Returning to the optimality conditions, we can rewrite them as a two-point boundary value problem.

$$
\begin{aligned}
\dot{x}_1 &= x_2 &&, \quad x_1(0) \text{ given}, \quad x_1(t_f) \text{ given} \\
\dot{x}_2 &= -\operatorname{sign}(\lambda_2) &&, \quad x_2(0) \text{ given}, \quad x_2(t_f) \text{ given} \\
\dot{\lambda}_1 &= 0 &&, \quad \lambda_1(0) \text{ unknown} \\
\dot{\lambda}_2 &= -\lambda_1 &&, \quad \lambda_2(0) \text{ unknown}
\end{aligned}
$$

See that the initial costates are unknown. Also, the final time is unknown. Thus, there are three unknowns. Fortunately, we will always have the right number of equations to resolve the unknowns.

$$x_1(t_f) = 0, \quad x_2(t_f) = 0$$

$$H_f = \lambda_0 + \lambda_{1f} x_{2f} + \lambda_{2f} u_f = 0.$$

The optimal control problem has been reduced to a root solving problem. Guess $\lambda_1(0)$, $\lambda_2(0)$, & t_f & use Newton's Method to iterate & satisfy the 3 equations.

An Approximate Shooting Method: Unfortunately, the above problem is non-smooth because of the $\text{sign}(\lambda_2)$ term.

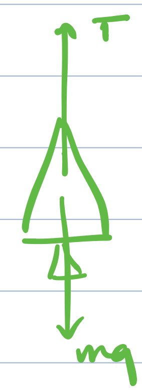
The problem can be approximated in a smooth way using the tanh function. In fact, $\lim_{\gamma \to \infty} \tanh(\gamma \lambda_2) = \text{sign}(\lambda_2)$.

Thus, the smooth boundary value problem is:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\tanh(\gamma \lambda_2)$$
$$\dot{\lambda}_1 = 0$$
$$\dot{\lambda}_2 = -\lambda_1$$

with all other constraints the same.

Terminal Descent Phase: Let's now look at a variation of the above problem which has gravity and mass dynamics. Consider a lunar lander in the vertical terminal descent phase.



The equations of motion are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -g + T/m$$
$$\dot{m} = -\alpha T$$

The objective is to minimize the flight time, and the thrust is bounded by T_{max}. Starting at some altitude + downward velocity, the vehicle must land on the surface with zero velocity.

The Hamiltonian for this problem is

$$H = \lambda_0 + \lambda_1 x_2 + \lambda_2 \left(-g + T/m\right) + \lambda_3 \left(-\alpha T\right)$$

The costate equations and transversality conditions are

$$\dot{\lambda}_1 = -\partial H / \partial x_1 = 0 \quad , \quad \lambda_{1f} = \nu_1$$

$$\dot{\lambda}_2 = -\partial H / \partial x_2 = -\lambda_1 \quad , \quad \lambda_{2f} = \nu_2$$

$$\dot{\lambda}_3 = -\partial H / \partial m = \frac{\lambda_2 T}{m^2} \quad , \quad \lambda_{3f} = 0$$

The pointwise minimum condition is

$$T \in \underset{0 \le w \le T_{max}}{\arg\min} \left(\frac{\lambda_2}{m} - \alpha \lambda_3 \right) w$$

$$= \begin{cases} 0 & \lambda_2/m - \alpha \lambda_3 > 0 \\ T_{max} & \lambda_2/m - \alpha\lambda_3 < 0 \\ \text{singular} , & \lambda_2/m - \alpha \lambda_3 \equiv 0 \end{cases}$$

We will now investigate the singular case. Suppose that

$$\lambda_2/m - \alpha\lambda_3 \equiv 0 \qquad \text{on some interval } [t_1, t_2]$$

$$\Rightarrow \dot{\lambda}_2 - \alpha\dot{m}\lambda_3 - \alpha m\dot{\lambda}_3 \equiv 0$$

$$\Rightarrow -\lambda_1 + \alpha T\underbrace{(\alpha\lambda_3 - \lambda_2/m)}_{0} \equiv 0$$

$$\Rightarrow \lambda_1 \equiv 0.$$

Thus, λ_1 is zero for all time since $\dot{\lambda}_1 = 0$. Furthermore, λ_2 is constant. Let's look at 3 cases.

1) Suppose $\lambda_2 = 0$. Then $\lambda_3 = 0$ on $[t_1, t_2]$ since $\lambda_2/m - \alpha\lambda_3 = 0$. Furthermore, $\dot{\lambda}_3 = 0$ everywhere such that $\lambda_3 = 0$ everywhere. At the final time, the Hamiltonian is zero causing $\lambda_0 = 0$. This violates non-triviality. Thus, $\lambda_2 \neq 0$.

2) Suppose $\lambda_2 > 0$. Then $\lambda_3 > 0$ since $\lambda_2/m - \alpha\lambda_3 = 0$. Also, $\dot{\lambda}_3 > 0$. A function that is positive & increasing cannot terminate at zero ($\lambda_{3f} = 0$). Thus, $\lambda_2 \not> 0$.

3) Suppose $\lambda_2 < 0$. Then $\lambda_3 < 0$ and $\dot{\lambda}_3 < 0$. Thus, it is impossible for $\lambda_{3f} = 0$. Thus, $\lambda_2 \nless 0$.

To summarize, we assumed singularity and then arrived at various impossibilities.

We now know that

$$T = \begin{cases} 0, & \lambda_2/m - \alpha\lambda_3 > 0 \\ T_{max}, & \lambda_2/m - \alpha\lambda_3 < 0. \end{cases}$$

We also know that the final phase of flight must be thrusting, or else the vehicle will not land with zero velocity.

As such, we "expect" the optimal solution to be a coasting phase (no thrust) followed by a powered phase (thrust).

This is shown in the 1964 paper by Meditch.

Singular Minimum Time Control

We will continue our investigation of minimum time problems by looking at the rendezvous of spacecraft in LEO.

Consider the relative motion of 2 vehicles described by the planar CWH equations.

$$\dot{x} = Ax + bu \quad , \quad A \text{ is } 4\times4, \; b \text{ is } 4\times1.$$

The goal is to drive the system from an initial state to the origin in minimum time w/ bounded control.

Note that I've written the control influence matrix as lowercase b indicating that the control is a scalar. An immediate question is: can a 2-d system be controlled by a single control?

To answer this question, we need to look at the controllability matrix.

$$C = [b, Ab, A^2b, A^3b], \quad \text{which is } 4\times4.$$

If this matrix is full row rank (4), then the system is controllable.

Let's first assume that $b = [0,0,1,0]^T$. That is, there is control on in the local vertical direction. Then,

$$\text{rank}(C) = 3,$$

and the system is not controllable.

Now, assume that $b = [0,0,0,1]^T$. Then,

$$\text{rank}(C) = 4,$$

and the system is controllable. Now that we know the system can be controlled, let's analyze optimal solutions. As always, we begin by writing the Hamiltonian & endpoint functions.

$$H = \lambda_0 + \lambda^T(Ax + bu)$$
$$G = \nu^T(x_f - 0)$$

The costate & transversality conditions are

$$\dot{\lambda} = -A^T\lambda, \quad \lambda_f = \nu, \quad H_f = 0$$

The pointwise minimum condition is

$$u \in \underset{-1 \le w \le 1}{\arg\min}\ \lambda^T b w = \begin{cases} -1, & \lambda^T b > 0 \\ +1, & \lambda^T b < 0 \\ \text{sing}, & \lambda^T b \equiv 0 \end{cases}$$

Let's see if the singular case can occur, i.e., is it possible for $\lambda^T b$ to be zero on a non-trivial interval of time?

Suppose $\lambda^T b \equiv 0$ on an interval $[t_1, t_2]$. Then, by differentiating, we get

$$\begin{aligned} \lambda^T b &= 0 \\ \dot{\lambda}^T b &= -\lambda^T A b = 0 \\ -\dot{\lambda}^T A b &= +\lambda^T A^2 b = 0 \\ \dot{\lambda}^T A^2 b &= -\lambda^T A^3 b = 0 \end{aligned}$$

We can stack these up in matrix form

$$[b, Ab, A^2 b, A^3 b]^T \lambda = 0$$

$$\Rightarrow C^T \lambda = 0$$

Note that if C is full row rank, then C^T is full column rank. That is, its null space is trivial and the only solution is $\lambda \equiv 0$.

If $\lambda = 0$ anywhere, then it is equal to zero everywhere since it is the solution of a homogenous ODE.

At the final time, $H = \lambda_0 + \cancel{\lambda^T}^{\,0}(Ax+bu) = 0 \;\Rightarrow\; \lambda_0 = 0$.
This violates non-triviality. Thus, singular solutions cannot occur and the optimal control is bang-bang.

How would things have changed if the system had two controls?

$$\dot{x} = Ax + b_1u_1 + b_2u_2$$

The analysis would be very similar, but we would now require both controllability matrices

$$C_1 = [\,b_1,\ Ab_1,\ A^2b_1,\ A^3b_1\,]$$

$$C_2 = [\,b_2,\ Ab_2,\ A^2b_2,\ A^3b_2\,]$$

to be full row rank. This is left as an exercise for you!

Let's now consider the rendezvous of three vehicles. The new system dynamics $\bar{A}$ & $\bar{B}$ are

$$\bar{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \bar{b}_1 = \begin{pmatrix} b \\ 0 \end{pmatrix}, \quad \bar{b}_2 = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

where A & b are the same as above. By defining

$$\bar{B} = [\bar{b}_1, \bar{b}_2]$$

it is a simple matter to show that

$$\bar{C} = [\bar{B}, \bar{A}\bar{B}, \dots, \bar{A}^{7}\bar{B}] \text{ is full row rank}$$

but $\bar{C}_1 = [\bar{b}_1, \bar{A}\bar{b}_1 \dots]$ is not
and $\bar{C}_2 = [\bar{b}_2, \bar{A}\bar{b}_2, \dots]$ is not.

As a result, we cannot rule out singular solutions.
Theorem 6.5 in the book by Athans & Falb tells us more: solutions to this problem <u>must</u> be singular!

The book by Athans & Falb also has a nice discussion on existence & uniqueness.

Let's suppose for concreteness that u_2 is singular. Just because it is singular doesn't mean that $u_2 \notin \{-1, 1\}$. Singular solutions can also be bang-bang.

In fact, LaSalle has a famous theorem known as the "bang-bang" theorem: If any solution exists, then a bang-bang solution also exists.

How you find the singular solutions or this magical bang-bang solution isn't so obvious. Maybe this is why so many authors ignore them!

My "Minimum Time Rendezvous" paper from 2014 provides one way of finding the bang-bang solution.

How can we find the singular solution(s)? A typical approach is to differentiate the switching function until the control appears, and then solve for it.

$$\lambda^T b = 0$$

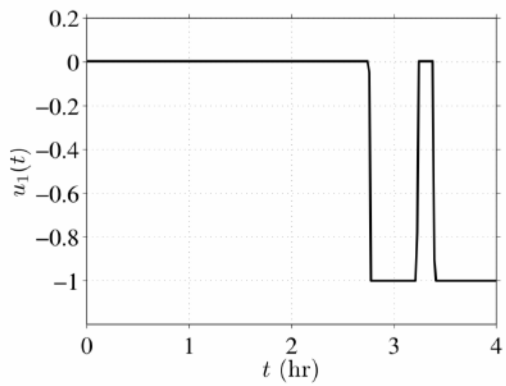
$$\dot{\lambda}^T b = -\lambda^T A^T b = 0$$

$$\vdots$$

For this problem, the control will never appear! Thus, we don't have an analytical way to solve for the control.

In other words, we don't know how to write $u = u(\lambda)$.

An effective approach here is to discretize and solve directly. The figure below shows a control for two vehicles to rendezvous. It is clearly bang-bang.



The figure below shows the optimal controls for 5 vehicles to rendezvous. The controls are singular but still bang-bang. They were found using Yalmip and a procedure described in my 2014 paper.

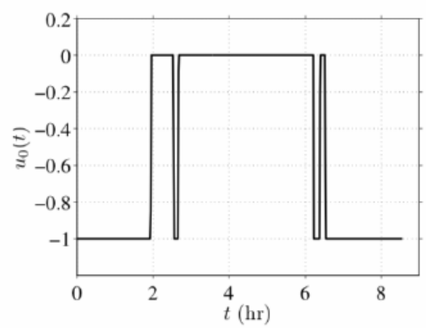


Figure 29: Target spacecraft control $u_0(t)$ with $M = 4$.

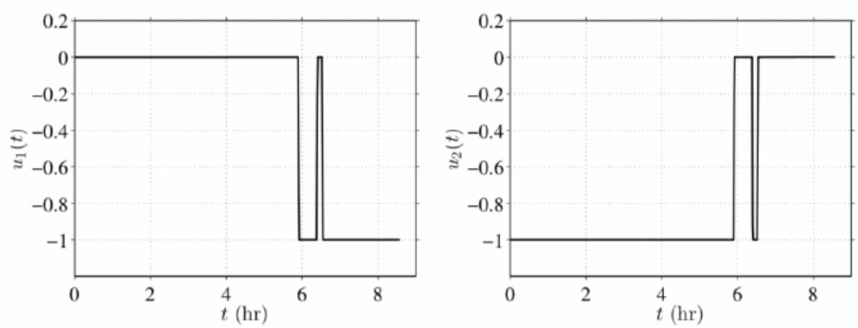


Figure 30: Chaser spacecraft control $u_1(t)$ and $u_2(t)$ with $M = 4$.

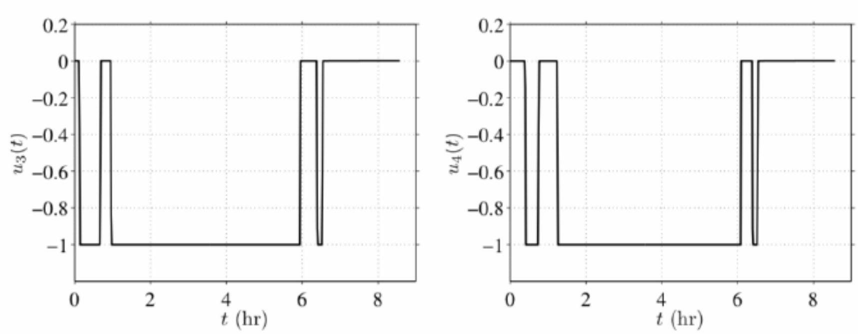


Figure 31: Chaser spacecraft control $u_3(t)$ and $u_4(t)$ with $M = 4$.

To summarize:

- Some problems do not have singular solutions. We often prove they don't using a controllability condition.
- Some problems do have singular solutions (e.g. the rendezvous of many spacecraft).
- When the solution is singular, the optimality conditions don't allow you to directly solve for the control. At times, they don't give any useful info! In these cases, direct methods are useful.

Goddard's Problem & Iterative Guidance Mode

We are going to study Goddard's problem, which was first proposed in 1919. It received significant attention in the 1950s & 1960s as it is an interesting optimal control problem

The problem is to determine the thrust profile to maximize the altitude of a rocket starting from rest on the surface.

The forces acting on the vehicle are thrust T, gravity g (which we assume constant for simplicity only), and drag $D(v,h)$ (which may depend on the speed and altitude.

The states of the system are altitude h, speed v, and mass m.

The thrust magnitude is bounded by $0 \le T \le T_{max}$.

The problem is:

$$\max \quad h(t_f)$$

$$\text{s.t.} \quad \dot{h} = v, \quad h(0) = 0$$

$$\dot{v} = T/m - D(v,h)/m - g, \quad v(0) = 0$$

$$\dot{m} = -\alpha T, \quad m(0) \text{ given}, \quad m(t_f) \text{ given as } \bar{m}_f$$

$$0 \le T \le T_{max}$$

Analysis of the problem begins by forming the Hamiltonian & endpoint functions.

$$H = \lambda_1 v + \lambda_2 \left(T/m - D/m - g \right) - \lambda_3 \alpha T$$

$$G = \lambda_0 h_f + \upsilon \left(m_f - \bar{m}_f \right)$$

The costate & transversality conditions are

$$\dot{\lambda}_1 = \frac{\lambda_2}{m} \frac{\partial D}{\partial h} \quad , \quad \lambda_{1_f} = \lambda_0$$

$$\dot{\lambda}_2 = -\lambda_1 + \frac{\lambda_2}{m} \frac{\partial D}{\partial v} \quad , \quad \lambda_{2_f} = 0$$

$$\dot{\lambda}_3 = \frac{\lambda_2 T}{m^2} - \frac{\lambda_2 D}{m^2} \quad , \quad \lambda_{3_f} = \upsilon$$

$H_f = 0$ (since the Hamiltonian is time invariant, we also know that $H = 0$ at all times)

The pointwise maximum condition is

$$T \in \underset{0 \le \omega \le T_{max}}{\arg\max} \left(\frac{\lambda_2}{m} - \alpha \lambda_3 \right) \omega$$

Note that we are using arg <u>max</u> since it is a maximization problem.

There are then 3 cases.

$$T = \begin{cases} T_{max} , & \lambda_2/m - \alpha\lambda_3 > 0 \\ 0 , & \lambda_2/m - \alpha\lambda_3 < 0 \\ \text{singular} , & \lambda_2/m - \alpha\lambda_3 = 0 \end{cases}$$

Let's explore the singular case. Along a singular arc (where $\lambda_2/m - \alpha\lambda_3 = 0$ for a non-trivial interval of time), we must have

$$\phi = \lambda_2 - \alpha m \lambda_3 = 0$$

$$\dot{\phi} = \dot{\lambda}_2 - \alpha \dot{m} \lambda_3 - \alpha m \dot{\lambda}_3 = 0$$

$$= -\lambda_1 + \frac{\lambda_2}{m}\frac{\partial D}{\partial v} + \alpha^2 T \lambda_3 - \alpha m \left(\frac{\lambda_2}{m^2} T - \frac{\lambda_2}{m^2} \right) D$$

$$= -\lambda_1 + \left(\frac{\partial D}{\partial v} + \alpha D \right) \frac{\lambda_2}{m} + \underbrace{\alpha\left(\alpha\lambda_3 - \lambda_2/m\right)T}_{0}$$

$$\Rightarrow \quad 0 = -m\lambda_1 + \left(\frac{\partial D}{\partial v} + \alpha D \right) \lambda_2$$

Continuing on, we differentiate this w.r.t. time and we get

$$T = D + mg + \frac{m}{D + 2c\frac{\partial D}{\partial v} + c^2 \frac{\partial^2 D}{\partial v^2}} \left[-g\left(D + c\frac{\partial D}{\partial v}\right) + c(c-v)\frac{\partial D}{\partial h} - vc^2 \frac{\partial^2 D}{\partial v \partial h} \right]$$

where $c = 1/\alpha$.

Thus, by differentiating the switching function twice, we found that the thrust would have to satisfy the above expression.

In this problem we could not rule out singular solutions. However, for a portion of the solution to be singular, we must have

$$H = 0, \quad \phi = 0, \quad \text{and} \quad \dot{\phi} = 0.$$

In matrix form, we must have

$$\left[\begin{array}{c|c|c} \frac{1}{m}(T-D) - g & v & -\alpha T \\ \frac{1}{\alpha} & 0 & -m \\ \frac{\partial D}{\partial v} + \alpha D & -m & 0 \end{array}\right] \begin{pmatrix} \lambda_2 \\ \lambda_1 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If the matrix is full rank, then $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Note that the costate equations are homogenous such that if they are zero somewhere, then they are zero everywhere. At the final time $\lambda_{1f} = \lambda_0$. But λ_0 cannot be zero since this would violate non-triviality. Thus, the above matrix cannot be full rank.

Computing the determinant & setting it to zero gives

$$D + mg - \alpha v D - v \frac{\partial D}{\partial v} = 0,$$

which must hold along any singular arc. This equation is sometimes called the singular surface.

We won't show it analytically, but solutions to this problem are typically of the form

$$\begin{cases} T = T_{max} \\ T = \text{singular} \\ T = 0 \end{cases}$$

This type of sequence is called bang-singular-off.

The implementation strategy is the following:

- Apply maximum thrust until the determinant becomes zero.
- Switch to the singular thrust until burn-out.
- Coast to maximum altitude.

Minimum Time Orbit Injection

We now assume constant thrust acceleration $\Upsilon = T/m$ and consider a minimum time orbit injection. x is the range, u is the range rate, y is the altitude and v is the altitude rate. The optimal control problem is below.

$$
\begin{aligned}
\min \quad & t_f \\
\text{s.t.} \quad & \dot{x} = u, && x(0) = 0, && x(t_f) \text{ is free} \\
& \dot{u} = \Upsilon\cos\theta, && y(0) = 0, && u(t_f) = u_f \\
& \dot{y} = v, && z(0) = 0, && y(t_f) = y_f \\
& \dot{v} = \Upsilon\sin\theta - g, && v(0) = 0, && v(t_f) = 0
\end{aligned}
$$

The Hamiltonian + Endpoint functions are

$$H = \lambda_1 u + \lambda_2 v + \lambda_3 \Upsilon\cos\theta + \lambda_4(\Upsilon\sin\theta - g)$$

$$G = \lambda_0 t_f + \nu_1\big(u(t_f) - u_f\big) + \nu_2\big(y(t_f) - y_f\big) + \nu_3\big(v(t_f) - 0\big)$$

The costate and transversality conditions are

$$\dot{\lambda}_1 = 0 \,, \quad \lambda_{1_f} = 0 \qquad (\lambda_1 \text{ is zero})$$
$$\dot{\lambda}_2 = 0 \,, \quad \lambda_{2_f} = \nu_2 \qquad (\lambda_2 \text{ is constant})$$
$$\dot{\lambda}_3 = -\lambda_1 \,, \quad \lambda_{3_f} = \nu_1 \qquad (\lambda_3 \text{ is constant})$$
$$\dot{\lambda}_4 = -\lambda_2 \,, \quad \lambda_{4_f} = \nu_3 \qquad (\lambda_4 \text{ is linear})$$

$$H_f = -\lambda_0$$

The optimal control is given by

$$\theta(t) \in \operatorname{argmin} \; \lambda_3 T \cos\theta + \lambda_4 T \sin\theta$$

$$\Rightarrow \; -\lambda_3 \sin\theta + \lambda_4 \cos\theta = 0$$

$$\Rightarrow \; \tan\theta = \frac{\lambda_4}{\lambda_3} = \frac{-\lambda_2(t_f - t) + \lambda_{4_f}}{\lambda_3}$$

We see that $\tan\theta$ is a linear function of time.
We now write the tangent law as

$$\tan\theta = \tan\theta_0 - ct$$

Using θ as the independent variable, the state equations can be integrated to the final point:

$$u_f = \frac{T}{c} \log \left[\frac{\tan\theta_0 + \sec\theta_0}{\tan\theta_f + \sec\theta_f} \right]$$

$$v_f = \frac{T}{c} \left[\sec\theta_0 - \sec\theta_f \right] - g t_f$$

$$x_f = \frac{T}{c^2} \left\{ \sec\theta_0 - \sec\theta_f - \tan\theta_f \log \left[\frac{\tan\theta_0 + \sec\theta_0}{\tan\theta_f + \sec\theta_f} \right] \right\}$$

$$y_f = \frac{T}{2c^2} \left\{ (\tan\theta_0 - \tan\theta_f)\sec\theta_0 - (\sec\theta_0 - \sec\theta_f)\tan\theta_f - \log \left[\frac{\tan\theta_0 + \sec\theta_0}{\tan\theta_f + \sec\theta_f} \right] \right\} - \frac{1}{2} g t_f^2$$

Note that $c = \dfrac{\tan\theta_0 - \tan\theta_f}{t_f}$.

As such, there are three unknowns in the above equations. These are θ_0, θ_f, and t_f.
We also have three boundary conditions: y_f, u_f, + v_f.
The three equations can be solved iteratively.

This idea was used to develop the "Iterative Guidance Mode" or IGM for the Saturn V ascent guidance.

To reduce numerical complexity in the solution process, or to facilitate an initial guess, one may make the following 1st-order approximation.

$$\theta = \theta_0 - ct$$

Powered Ascent Guidance

We previously derived the linear tangent law and its use in the iterative guidance mode for Saturn V. A different concept was used for the shuttle known as powered explicit guidance – or PEG. PEG has also been discussed as an option for future lunar missions.

In these notes, we'll look at a recently improved version of PEG developed by David Hull & myself. It was published in the Journal of Guidance, Control, & Dynamics in 2012 as "Optimal Solutions for Quasiplanar Ascent over a Spherical Moon."

To begin, we'll write the equations of motion as

$$
\begin{aligned}
\dot{x} &= u r_m / r, & \dot{u} &= \tau \cos\theta \cos\psi + (uw/r)\tan(z/r_m) - uv/r \\
\dot{y} &= v, & \dot{v} &= \tau \sin\theta - g + u^2/r + w^2/r \\
\dot{z} &= w, & \dot{w} &= \tau \cos\theta \sin\psi - (u^2/r)\tan(z/r_m) - vw/r
\end{aligned}
$$

The radius of the moon is r_m. The radial position of the vehicle is $r = r_m + y$. x is the curvilinear in-plane distance, and y is the in-plane altitude. z is the out-of-plane curvilinear distance. $u, v, \& \ w$ are the velocities. τ is the thrust to mass T/m. $\theta \ \& \ \psi$ are thrust angles.

We will now make several assumptions:

- out-of-plane motion is small (hence quasi-planar)
- $y/r_m \ll 1$ s.t. $r \cong r_m$ and g is constant
- $uv/r_m \ll \mathcal{T}$

As a result, the equations reduce to

$$
\begin{aligned}
\dot{x} &= u, & \dot{u} &= \mathcal{T}\cos\theta\cos\psi \\
\dot{y} &= v, & \dot{v} &= \mathcal{T}\sin\theta - g_m + u^2/r_m \qquad \text{(EOMs)} \\
\dot{z} &= w, & \dot{w} &= \mathcal{T}\cos\theta\sin\psi
\end{aligned}
$$

For constant thrust, minimizing fuel consumption is the same as minimizing flight time. Thus, we have the following optimal control problem.

$$\min \quad t_f$$

s.t. EOMs

initial states specified

final states specified except for x_f

The Hamiltonian & endpoint functions are

$$H = \lambda_1 u + \lambda_2 v + \lambda_3 w + \lambda_4 \Upsilon \cos\theta \cos\psi$$
$$+ \lambda_5 (\Upsilon \sin\theta - g_m + u^2/r_m) + \lambda_6 \Upsilon \cos\theta \sin\psi$$
$$G = \lambda_0 t_f + \nu_2 (y_f - \bar{y}_f) + \nu_3 (z_f - \bar{z}_f)$$
$$+ \nu_4 (u_f - \bar{u}_f) + \nu_5 (v_f - \bar{v}_f) + \nu_6 (w_f - \bar{w}_f)$$

The costate & transversality conditions are

$$\dot{\lambda}_1 = 0 \quad , \quad \dot{\lambda}_4 = -2\lambda_5 u / r_m \quad , \quad \lambda_{1f} = 0$$
$$\dot{\lambda}_2 = 0 \quad , \quad \dot{\lambda}_5 = -\lambda_2$$
$$\dot{\lambda}_3 = 0 \quad , \quad \dot{\lambda}_6 = -\lambda_3$$
$$H_f = -\lambda_0$$

We see that $\lambda_1, \lambda_2,$ & λ_3 are constants. Furthermore,

$$\lambda_5 = -\lambda_2 t + c_2 \quad , \quad \lambda_6 = -\lambda_3 t + c_3 .$$

Because there are no control constraints, the pointwise minimum condition leads to

$$\partial H / \partial \theta = -\lambda_4 \Upsilon s\theta c\psi + \lambda_5 \Upsilon c\theta - \lambda_6 \Upsilon s\theta s\psi = 0$$
$$\partial H / \partial \psi = -\lambda_4 \Upsilon c\theta s\psi + \lambda_6 \Upsilon c\theta c\psi = 0$$

The above equations can be solved (see the paper for details)

$$\sin\psi = \frac{-\lambda_6}{\sqrt{\lambda_4^2+\lambda_6^2}}, \quad \cos\psi = \frac{-\lambda_4}{\sqrt{\lambda_4^2+\lambda_6^2}} \qquad (*)$$

$$\sin\theta = \frac{-\lambda_5}{\sqrt{\lambda_4^2+\lambda_5^2+\lambda_6^2}}, \quad \cos\theta = \frac{\sqrt{\lambda_4^2+\lambda_6^2}}{\sqrt{\lambda_4^2+\lambda_5^2+\lambda_6^2}}$$

It is expected that both thrust angles will be small such that

$$\left(\lambda_6/\lambda_4\right)^2 \ll 1 \quad \text{and} \quad \left(\lambda_5/\lambda_4\right)^2 \ll 1.$$

Under these assumptions the controls are given by

$$\sin\psi = \lambda_6/\lambda_4, \quad \cos\psi = 1, \quad \sin\theta = \lambda_5/\lambda_4, \quad \cos\theta = 1.$$

The resulting boundary value problem is

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{z} = w, \quad \dot{u} = \tau$$

$$\dot{v} = \tau/\lambda_4\left(-\lambda_2 t + c_2\right) - g_m + u^2/r_m$$

$$\dot{w} = \tau/\lambda_4\left(-\lambda_3 t + c_3\right), \quad \dot{\lambda}_4 = -2\left(-\lambda_2 t + c_2\right)u/r_m$$

There are 6 unknowns: $\lambda_2, c_2, \lambda_3, c_3, \lambda_4(0), t_f$.
We have 5 known final conditions plus $H_f = -\lambda_0$. It will be shown that all unknowns can be divided by $\lambda_4(0)$ eliminating the need for $H_f = -\lambda_0$.

The solution process begins by integrating the $\dot{u}$, $\dot{x}$, + $\dot{\lambda}_4$ equations. For operation at constant thrust

$$T = \beta V_e$$

where β is the propellant mass flow rate + V_e is the exhaust velocity. Hence, the mass as a function of time is

$$m = m_0 - \beta t$$

Thus, the thrust to mass ratio is

$$T/m = \gamma = \beta V_e / (m_0 - \beta t) = -V_e / (t - \alpha)$$

where $\alpha = m_0/\beta$. We can now integrate u + x directly

$$u = u_0 - V_e \ln(1 - t/\alpha)$$
$$x = x_0 + u_0 t + V_e(\alpha - t)\ln(1 - t/\alpha) + V_e t$$

Now that we know u, we can substitute in the $\dot{\lambda}_y$ equation & integrate.

initial value for λ_y.

$$\lambda_y = c_1 - (2u_0/r_m)(-\lambda_2 t^2/2 + c_2 t)$$

$$-(v_e \lambda_2/2r_m)\left[2(t^2-\alpha^2)\ln(1-t/\alpha) - 2\alpha t - t^2\right]$$

$$-(2v_e c_2/r_m)\left[(\alpha - t)\ln(1-t/\alpha) - t\right]$$

Dividing through by c_1 gives us the bar variables.

$$\bar{\lambda}_y = 1 - (2u_0/r_m)(-\bar{\lambda}_2 t^2/2 + \bar{c}_2 t)$$

$$-(v_e \bar{\lambda}_2/2r_m)\left[2(t^2-\alpha^2)\ln(1-t/\alpha) - 2\alpha t - t^2\right]$$

$$-(2v_e \bar{c}_2/r_m)\left[(\alpha - t)\ln(1-t/\alpha) - t\right]$$

We see that $\bar{\lambda}_y$ is a function of $\bar{\lambda}_2$, $\bar{c}_2$, and t. The remaining equations of motion can be rewritten as

$$\dot{y} = v \quad , \quad \dot{v} = \tau(\bar{\lambda}_5/\bar{\lambda}_y) - g_m + \left[u_0 - v_e \ln(1-t/\alpha)\right]^2/r_m$$

$$\dot{z} = w \quad , \quad \dot{w} = \tau(\bar{\lambda}_6/\bar{\lambda}_y)$$

Some parts of these equations can be integrated analytically, while others require numerical integration. Generally, solution of the above differential equations can be written in terms of the following (which are integrals):

$$\underbrace{J'(t_f, \bar{\lambda}_2, \bar{c}_2) \;,\; L'(t_f, \bar{\lambda}_2, \bar{c}_2) \;,\; Q'(t_f, \bar{\lambda}_2, \bar{c}_2) \;,\; S'(t_f, \bar{\lambda}_2, \bar{c}_2)}_{\text{"modified" thrust integrals}}$$

$$\underbrace{F(t_f) \quad \text{and} \quad G(t_f)}_{\text{centrifugal integrals}}$$

See the paper for their definitions. Then,

$$v = v_0 - \bar{\lambda}_2 \hat{J}' + \bar{c}_2 \hat{L}' - g_m t + \hat{F}$$

$$y = y_0 + v_0 t - \bar{\lambda}_2 \hat{Q}' + \bar{c}_2 \hat{S}' - g_m t^2/2 + \hat{G}$$

$$w = w_0 - \bar{\lambda}_3 \hat{J}' + \bar{c}_3 \hat{L}'$$

$$z = z_0 + w_0 t - \bar{\lambda}_3 \hat{Q}' + \bar{c}_3 \hat{S}'$$

where the hats $(\wedge)$ denote that the integrals are evaluated at time t.

Evaluating these at the final time gives us 5 equations to solve for the 5 unknowns. Define the following:

$$V_x = u_f - u_0 \quad , \quad V_y = v_f - v_0 + g_m t_f - F$$

$$Y = y_f - y_0 - v_0 t_f + g_m t_f^2/2 - G \quad , \quad V_z = w_f - w_0$$

$$Z = z_f - z_0 - w_0 t_f$$

Then, the u equation can be used to solve for t_f.

$$t_f = \alpha\left[1 - e^{-V_x/V_e}\right]$$

With t_f known, the v + y equations can be solved iteratively for $\bar{\lambda}_2$ and $\bar{c}_2$.

$$V_y = -\bar{\lambda}_2 J' + \bar{c}_2 L'$$
$$Y = -\bar{\lambda}_2 Q' + \bar{c}_2 S'$$

With $\bar{\lambda}_2$ and $\bar{c}_2$ known, the w + z equations can be solved analytically for $\bar{\lambda}_3$ and $\bar{c}_3$.

$$\bar{\lambda}_3 = \left(V_z S' - z L'\right) / \left(L'Q' - J'S'\right)$$
$$\bar{C}_3 = \left(V_z Q' - z J'\right) / \left(L'Q' - J'S'\right)$$

Thus, we were able to solve for the 5 unknowns (3 analytically and 2 implicitly using iterations).

As a final step, we must calculate the controls. This is done using Eq (*) a few pages back (but using the bar quantities).

While the resulting solution is relatively simple, it is not completely analytical. Section 5 of the paper presents an approximation strategy (based on $\tilde{\lambda}_4 \approx 1$) that yields an analytical solution!

Continuous Thrust Orbit Transfers

Consider a spacecraft in a circular orbit. What is the largest circular orbit it can reach? The optimal control problem is:

$$\max \quad r(t_f) \quad , \quad t_f \text{ is given}$$

$$\text{s.t.} \quad \dot{r} = u \quad , \quad r(0) = r_0$$

$$\dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} + \frac{T\sin\theta}{m_0 - \dot{m}t} \quad , \quad u(0) = 0$$

$$\dot{v} = -\frac{uv}{r} + \frac{T\cos\theta}{m_0 - \dot{m}t} \quad , \quad v(0) = \sqrt{\mu/r_0}$$

$$u(t_f) = 0 \quad , \quad v(t_f) = \sqrt{\mu/r_f}$$

The variables are:

- r is radial distance
- u is radial velocity
- v is tangential velocity
- m_0 is initial mass
- $\dot{m}$ is fuel burn rate
- θ is thrust angle
- T is thrust force

To solve the problem

$$H = \lambda_r u + \lambda_u \left(\frac{v^2}{r} - \frac{\mu}{r^2} + \frac{T \sin\theta}{m_0 - \dot{m} t} \right) + \lambda_v \left(\frac{-uv}{r} + \frac{T \cos\theta}{m_0 - \dot{m} t} \right)$$

$$G = \lambda_0 r_f + \nu_1 u_f + \nu_2 \left(v_f - \sqrt{\mu / r_f} \right)$$

The costate equations are

$$\dot{\lambda}_r = -\frac{\partial H}{\partial r} = -\lambda_u \left(\frac{-v^2}{r^2} + \frac{2\mu}{r^3} \right) - \lambda_v \left(\frac{uv}{r^2} \right)$$

$$\dot{\lambda}_u = -\frac{\partial H}{\partial u} = -\lambda_r + \lambda_v \frac{v}{r}$$

$$\dot{\lambda}_v = -\frac{\partial H}{\partial v} = -\lambda_u \frac{2v}{r} + \lambda_v \frac{u}{r}$$

The transversality conditions are

$$\lambda_{r_f} = \frac{\partial G}{\partial r_f} = \lambda_0 + \frac{\nu_2 \sqrt{\mu}}{2 r_f^{3/2}}$$

$$\lambda_{u_f} = \nu_1$$

$$\lambda_{v_f} = \nu_2$$

The optimal control maximizes the Hamiltonian

$$\theta(t) \in \operatorname{argmax} \; \lambda_u \sin\theta + \lambda_v \cos\theta$$

$$\Rightarrow \quad \tan\theta = \frac{\lambda_u}{\lambda_v}$$

It is impossible to integrate the resulting equations analytically. We will use a shooting method.

Note that there are 6 ODEs (3 states and 3 costates). The initial conditions for the states are known. Guess the initial conditions for the costates. All six equations can now be integrated together to the final time. Check if

$$u(t_f) = 0 \quad , \quad v(t_f) = \sqrt{\mu / r_f}$$

and $$\lambda_{r_f} = \lambda_0 + \frac{\lambda_{v_f}\sqrt{\mu}}{2 r_f^{3/2}}$$

If so, you're done. Otherwise, iterate using Newton's method.

Shooting method

We have seen that the optimality conditions of optimal control involve two sets of differential equations – the state equations & the costate equations.

In a typical case, we get the following system.

$$\begin{aligned} \dot{x} &= f(x,u), \quad x(t_0) = x_0, \quad x(t_f) = x_f \\ \dot{\lambda} &= -\nabla_x f \cdot \lambda, \end{aligned}$$

Moreover, the pointwise minimum condition allows us to write the control as a function of λ, i.e., $u = u(\lambda)$. Thus, we have a system of $2n$ coupled differential equations with split boundary conditions.

$$\begin{aligned} \dot{x} &= f(x,\lambda), \quad x(t_0) = x_0, \quad x(t_f) = x_f \\ \dot{\lambda} &= -\nabla_x f(x,\lambda)\lambda, \end{aligned}$$

A set of differential equations where all initial conditions are known is called an initial value problem (IVP). Differential equations where some initial conditions & some final conditions are known form a two-point boundary value problem (TPBVP).

How can you solve an IVP? IVPs are easy to solve. Give the problem to an integrator such as MATLAB's ode45.

How can you solve a TPBVP? TPBVPs are considerably more difficult as they are similar to solving nonlinear equations. MATLAB has a built-in solver BVP4C, but we will often need more flexibility than afforded by it.

Here is a general approach:

- Guess the unknown intial conditions.
- Integrate to the final time.
- Check if the final conditions are met.
- If not, update the guess & repeat.

This procedure can be automated in MATLAB using

- ode45 for integration
- fsolve for updating (i.e., solving the equations)

Analytical Example: Solve the following problem.

$$\dot{x} = -x + \lambda\,, \qquad x(0) = 0, \quad x(1) = 1.$$
$$\dot{\lambda} = 1$$

Integrating the λ equation gives

$$\lambda = \lambda_0 + t$$

Substituting into the state equation gives

$$\dot{x} = -x + \lambda_0 + t$$

Integrating this equation yields

$$x = (1-\lambda_0)e^{-t} - (1-\lambda_0) + t$$

At the final time, we get

$$1 = (1-\lambda_0)e^{-1} - (1-\lambda_0) + 1$$

$$\Rightarrow \quad 0 = (e^{-1}-1)(1-\lambda_0) \quad \Rightarrow \quad \lambda_0 = 1$$

Let's check that the x equation goes from 0 to 1.

$$x = (1-1)e^{-t} - (1-1) + t = t$$

Indeed $x(0)=0$ and $x(1)=1$. Thus, we've solved the TPBVP by setting $\lambda_0 = \lambda(0) = 1$.

Another Example: Solve the following problem.

$$\begin{aligned} \dot{x} &= x - \lambda t \\ \dot{\lambda} &= -\lambda \end{aligned} \quad , \quad x(0) = x_0, \; x(1) = x_1$$

Solving the λ equation first gives

$$\lambda = \lambda_0 e^{-t}$$

Substituting into the x equation gives

$$\dot{x} = x - \lambda_0 t e^{-t}$$

$$\Rightarrow \quad x = \frac{\lambda_0}{4}(2t+1)e^{-t} + (x_0 - \lambda_0/4)e^{t}$$

Evaluating at the final time gives

$$x_1 = \frac{3}{4}\lambda_0 e^{-1} + x_0 e^1 - \frac{\lambda_0}{4} e^1$$

$$\Rightarrow \left(x_1 - x_0 e\right) = \frac{1}{4}\left(3e^{-1} - e^1\right)\lambda_0$$

$$\Rightarrow \lambda_0 = \frac{4\left(x_1 - x_0 e\right)}{3e^{-1} - e^1}$$

Numerical Example: We now consider a more challenging problem. It is challenging because the ODEs are coupled and nonlinear.

$$\dot{x} = x^2 - \lambda t \quad , \quad x(0) = 0$$
$$\dot{\lambda} = -\lambda x \quad , \quad \lambda(1) \cong 1.1785$$

Set this up in MATLAB using ode45 + fsolve.

Analytic Guidance Strategies for Landing

We'll look at some analytical & semi-analytical approaches to landing guidance. To arrive at simple solutions, we have to simplify the dynamic model.

One such approach was developed by Chris D'Souza in 1997. His paper is called "An optimal guidance law for planetary landing." We'll follow his approach.

Assume that gravity is constant, aerodynamic forces are negligible, mass dynamics are unimportant, and there are no control constraints. The resulting equations of motion are:

$$\dot{x} = u \quad , \quad \dot{u} = a_x \qquad \text{downrange}$$
$$\dot{y} = v \quad , \quad \dot{v} = a_y \qquad \text{crossrange}$$
$$\dot{z} = w \quad , \quad \dot{w} = a_z + g \qquad \text{altitude}$$

As an objective, he considers the weighted "time-energy" function

$$J = \Gamma t_f + \frac{1}{2}\int_{t_0}^{t_f} a_x^2 + a_y^2 + a_z^2 \, dt$$

Γ is a scalar weight. For small values of Γ, we expect longer flight times & smaller control values. For large values of Γ, we expect shorter flight times & larger control values.

The paper claims "a minimum time to landing can be obtained quite easily by setting Γ to a large positive number." Do you think a minimum time solution exists for a problem w/o control constraints?

To analyze the problem, we write the Hamiltonian & endpoint functions:

$$H = \frac{1}{2}\left(a_x^2 + a_y^2 + a_z^2\right) + \lambda_x u + \lambda_y v + \lambda_z w$$
$$+ \lambda_u a_x + \lambda_v a_y + \lambda_w (a_z + g)$$

$$G = \Gamma t_f + \underbrace{\nu_x x_f + \nu_y y_f + \nu_z z_f + \nu_u u_f + \nu_v v_f + \nu_w w_f}_{\text{landing at the origin w/ zero speed.}}$$

What happened to λ_0? D'souza is ignoring it by assuming $\lambda_0 = 1$. We should not do this. As an exercise, explore the $\lambda_0 = 0$ case.

The costate & transversality conditions are

$$\dot{\lambda}_x = 0 \qquad \lambda_{x_f} = \nu_x$$
$$\dot{\lambda}_y = 0 \qquad \lambda_{y_f} = \nu_y$$

$$\dot{\lambda}_z = 0 \qquad \lambda_{z_f} = \nu_z$$
$$\dot{\lambda}_u = -\lambda_x \qquad \lambda_{u_f} = \nu_u$$
$$\dot{\lambda}_v = -\lambda_y \qquad \lambda_{v_f} = \nu_v$$
$$\dot{\lambda}_w = -\lambda_z \qquad \lambda_{w_f} = \nu_w$$

We can easily integrate these equations. By defining $t_{go} = t_f - t$ (which is the amount of time remaining in the trajectory, they are

$$\lambda_x = \nu_x \quad , \quad \lambda_u = \nu_x t_{go} + \nu_u$$
$$\lambda_y = \nu_y \quad , \quad \lambda_v = \nu_y t_{go} + \nu_v$$
$$\lambda_z = \nu_z \quad , \quad \lambda_w = \nu_z t_{go} + \nu_w$$

The pointwise minimum condition is

$$a_x = \arg\min_{\sigma} \frac{1}{2}\sigma^2 + \lambda_u \sigma$$

$$a_y = \arg\min_{\sigma} \frac{1}{2}\sigma^2 + \lambda_v \sigma$$

$$a_z = \arg\min_{\sigma} \frac{1}{2}\sigma^2 + \lambda_w \sigma$$

Since all of the control accelerations are unconstrained, the minimizers can be found by setting the derivatives equal to zero. Thus,

$$a_x = -\lambda_u = -\nu_x t_{go} - \nu_u$$

$$a_y = -\lambda_v = -\nu_y t_{go} - \nu_v$$

$$a_z = -\lambda_w = -\nu_z t_{go} - \nu_w$$

Note that the control in each direction is a linear function of time since all of the ν's are constants. These functions can be substituted into the state equations & integrated to yield:

$$u = \frac{1}{2}\nu_x t_{go}^2 + \nu_u t_{go} \,, \quad x = -\frac{1}{6}\nu_x t_{go}^3 - \frac{1}{2}\nu_u t_{go}^2$$

$$v = \frac{1}{2}\nu_y t_{go}^2 + \nu_y t_{go} \,, \quad y = -\frac{1}{6}\nu_y t_{go}^3 - \frac{1}{2}\nu_v t_{go}^2$$

$$w = \frac{1}{2}\nu_z t_{go}^2 + \nu_w t_{go} - g t_{go} \,, \quad z = -\frac{1}{6}\nu_z t_{go}^3 - \frac{1}{2}\nu_w t_{go}^2 + \frac{1}{2} g t_{go}^2$$

Note that the minus signs appear in $x, y, + z$ because t_{go} is a negative function of time.

If we know our current state (u, v, w, x, y, z) and remaining flight time (t_{go}), then we can solve for all the ν's since they appear linearly in the above equations.

Once we know the ν's we can easily calculate the optimal accelerations a_x, a_y, & a_z.

$$a_x = \frac{-4u}{t_{go}} - \frac{6x}{t_{go}^2}$$

$$a_y = \frac{-4v}{t_{go}} - \frac{6y}{t_{go}^2}$$

$$a_z = \frac{-4w}{t_{go}} - \frac{6z}{t_{go}^2} - g$$

Note that as we approach t_f, t_{go} approaches zero causing the control accelerations to explode. One work-around here is to simply hold t_{go} constant once some minimal value is reached.

To this point, we've ignored calculation of the flight time. To find t_f, we need to use the other transversality condition $H_f = -\partial b / \partial t_f = -\Gamma$. Note, however, that in all of our analysis, we've assumed that $t_{go} \neq 0$. Thus, using this condition isn't too insightful at the moment.

An alternative is to use the fact that the Hamiltonian is also constant (since our problem is time-invariant). We won't go through all the details, but observe the following facts:

- $a_x^2 = \left(\frac{-4u}{t_{go}} - \frac{6x}{t_{go}^2}\right)^2 \sim \frac{1}{t_{go}^4}$
- $u = \frac{1}{2}\left(\frac{6u}{t_{go}^2} + \frac{12x}{t_{go}^3}\right)t_{go}^2 - \left(\frac{2u}{t_{go}} + \frac{6x}{t_{go}^2}\right)t_{go} \sim \frac{1}{t_{go}}$
- $\lambda_x u = v_x u \sim \frac{1}{t_{go}^4}$
- and similarly for other terms...

Thus, multiplying through by t_{go}^4 will result in a quartic equation, which can be solved analytically. According to D'Souza, that equation is

$$\left(\Gamma + \tfrac{1}{2}g^2\right)t_{go}^4 - 2\left(u^2 + v^2 + w^2\right)t_{go}^2 - 12\left(ux + vy + wz\right)t_{go} - 18\left(x^2 + y^2 + z^2\right) = 0$$

Of course, multiple solutions exist, and we should the least positive, real root. This completes the analysis for this particular guidance law.

There are infinitely many alternatives to the above guidance law. There are entire classes of laws dating back to the Apollo days. Ping Lu authored a paper linking many of these titled "The theory of fractional-polynomial powered descent guidance" in the Journal of Guidance, Control, & Dynamics in 2020.

In the absence of optimality, generating trajectories can be quite easy. (Think back to how we used polynomials to fit curves between our boundary conditions.) To see this, we'll work through Lu's first example.

Again, we assume a constant gravity field so that the equations of motion are

$$\dot{r} = v$$
$$\dot{v} = a + g$$

where $r, v, a, g \in \mathbb{R}^3$. We also continue to ignore mass dynamics, aerodynamic forces, & control constraints.

We use time-to-go as we did before $t_{go} = t_f - t$.

To achieve a simple guidance law, we assume a two-term parameterization, i.e., we specify a desired thrust acceleration of the form

$$a_d = c_1 \phi_1(t_{go}) + c_2 \phi_2(t_{go})$$

where $c_1, c_2 \in \mathbb{R}^3$ are constants. The ϕ functions are basis functions (functions we get to choose). We denote their first & second integrals as

$$\bar{\phi}_i(t_{go}) = \int_{t_{go}}^{0} \phi_i(\tau)\, d\tau$$

$$\hat{\phi}_i(t_{go}) = \int_{t_{go}}^{0} \bar{\phi}_i(\tau)\, d\tau .$$

We can then easily integrate the state equations to get desired velocity & position vectors.

$$v_d(t) = c_1 \bar{\phi}_1(t_{go}) + c_2 \bar{\phi}_2(t_{go}) - g\, t_{go}$$

$$r_d(t) = c_1 \hat{\phi}_1(t_{go}) + c_2 \hat{\phi}_2(t_{go}) + \tfrac{1}{2} g\, t_{go}^2$$

To track this trajectory, we consider the following feedback form

$$a(t) = a_d(t) - \beta_v(t_{go})\left[v(t) - v_d(t)\right]$$
$$- \beta_r(t_{go})\left[r(t) - r_d(t)\right]$$

where β_v and β_r are feedback gains that must be determined. Substituting in for a_d, v_d, + r_d gives

$$a = c_1\left(\phi_1 + \beta_v \bar{\phi}_1 + \beta_r \hat{\phi}_1\right) + c_2\left(\phi_2 + \beta_v \bar{\phi}_2 + \beta_r \hat{\phi}_2\right)$$
$$+ g\, t_{go}\left(\tfrac{1}{2}\beta_r t_{go} - \beta_v\right) - \beta_v v(t) - \beta_r r(t).$$

We now choose β_v and β_r s.t. the c_1 + c_2 coefficients go to zero.

$$\beta_v = \frac{\hat{\phi}_2 \phi_1 + \hat{\phi}_1 \phi_2}{\Delta}, \quad \beta_r = \frac{-\bar{\phi}_2 \phi_1 + \bar{\phi}_1 \phi_2}{\Delta}$$

$$\Delta = \hat{\phi}_1 \bar{\phi}_2 - \hat{\phi}_2 \bar{\phi}_1 \neq 0$$

With this selection, the acceleration becomes

$$a = g\, t_{go}\left(\tfrac{1}{2}\beta_r t_{go} - \beta_v\right) - \beta_v v(t) - \beta_r r(t)$$

which will guide the vehicle from its current state to the origin terminating with zero velocity.

Note that we never specified the basis functions ϕ_1 + ϕ_2. Now, let $\phi_1 = 1$ and $\phi_2 = t_{go}$. Then,

$$\beta_r = \frac{6}{t_{go}^2}, \quad \beta_v = \frac{4}{t_{go}}$$

and the guidance law is

$$a = \frac{2}{t_{go}} v(t) - \frac{6}{t_{go}^2}\left[r(t) + v(t) t_{go}\right] - g.$$

$$= \frac{-4v(t)}{t_{go}} - \frac{6r(t)}{t_{go}^2} - g$$

This particular guidance law is called the E-guidance law. Explicit
It was first derived by Cherry in 1964 though not in this way. The final form is the same as that of D'souza.

Ping Lu goes on to describe many other guidance laws. So please read his paper.

Computational Guidance Strategies for Landing

We previously investigated analytical strategies for landing. Such strategies required minimal computation but required numerous assumptions. We'll now weaken some of those assumptions, which will require us to do more computation. We will focus on convex optimization approaches since these have provable convergence in polynomial time.

Like last time, we will ignore aerodynamic forces and assume constant gravity. Unlike last time, we will consider mass dynamics and control constraints. Thus, the equations of motion are

$$\dot{r} = v, \quad \dot{v} = T/m + g$$
$$\dot{m} = -\alpha \| T \|$$
$$\| T \| \leq \rho$$

where r is the position, v is the velocity, m is the mass, and ρ is thrust magnitude bound.

A typical objective is to minimize fuel consumption, i.e.,

$$\min \quad J = \int_0^{t_f} \| T \| \, dt$$

to transfer the vehicle from its current state to a specified state.

As written, this problem is non-convex because

$$\dot{v} = T/m + g \quad \text{is nonlinear}$$

and $\dot{m} = -\alpha \|T\|$ is nonlinear.

To get around this, we introduce the following transformations.

$$u = T/m \,, \quad \sigma = \|T\|/m$$

The equations of motion are

$$\left.\begin{aligned} \dot{r} &= v \\ \dot{v} &= u + g \end{aligned}\right\} \text{which are now linear}$$

$$\dot{m} = -\alpha m \sigma \quad \} \text{ which remains nonlinear}$$

The mass equation can then be written as

$$\frac{\dot{m}}{m} = -\alpha\sigma \quad \Rightarrow \quad m(t) = m_0 \exp\left[-\alpha \int_0^t \sigma(\tau)\, d\tau\right]$$

We can see that minimizing the fuel (or maximizing the final mass) is equivalent to

$$\min \; J = \int_0^{t_f} \sigma(\tau)\, d\tau$$

We can then write $\|u\| = \sigma$ as an inequality $\|u\| \leq \sigma$ and it will naturally be satisfied since σ is being minimized.

We now return to our mass dynamics, which are non-convex. We can rigorously linearize them through another variable transformation. Let $z = \ln(m)$ such that

$$\dot{z} = \frac{\dot{m}}{m} = -\alpha\sigma$$

This is linear! Unfortunately, the control constraint is now non-convex since e^{-z} is non-convex in z.

$$\|T\| \leq \rho \longrightarrow m\sigma \leq \rho \longrightarrow \sigma \leq \rho e^{-z}$$

We are now forced to make some approximation. Any approximation should be conservative in the sense that the above constraint is satisfied.

One <u>overly</u> conservative approach is to use an upper bound on $m(t)$. One such upper bound is $z_0 = \ln(m_0)$. Then

$$\sigma \leq \frac{\rho}{m_0} = \rho e^{-z_0} \leq \rho e^{-z}$$

A better alternative is to approximate the non-linearity with a Taylor series centered at $\tilde{z}$ (a good $\tilde{z}$ is to be determined).

$$\rho e^{-z} \approx \rho e^{-\tilde{z}} - \rho e^{-\tilde{z}}(z-\tilde{z})$$

We can easily show that this linear approximation is conservative using the mean value theorem, which says there is a $\hat{z}$ s.t.

$$\rho e^{-z} = \rho e^{-\tilde{z}} - \rho e^{-\tilde{z}}(z-\tilde{z}) + \frac{1}{2}\rho e^{-\hat{z}}(z-\tilde{z})^2$$

Since the last term is non-negative, we conclude that

$$\rho e^{-\tilde{z}} - \rho e^{-\tilde{z}}(z-\tilde{z}) \leq \rho e^{-z}$$

As for $\tilde{z}$, we can provide a guess such as

$$\tilde{z}(t) = \begin{cases} \ln(m_0 - \alpha \rho t), & m_0 - \alpha \rho t \geq m_{dry} \\ \ln(m_{dry}) & \text{otherwise} \end{cases}$$

With this definition of $\tilde{z}(t)$, we know that

$$\tilde{z}(t) \leq z(t)$$

To summarize, we've transformed a non-convex problem into a convex form. The transformation is not exact since we made an approximation. However, the transformation is feasible since we ensured the approximation was conservative.

The resulting convex problem is stated below:

$$\min \int_0^{t_f} \sigma(\tau)\, d\tau$$

$$\begin{aligned}
\text{s.t.} \quad & \dot{r} = v \quad , \quad r_0 \text{ given} \quad , \quad r_f \text{ given} \\
& \dot{v} = u + g \quad , \quad v_0 \text{ given} \quad , \quad v_f \text{ given} \\
& \dot{z} = -\alpha \sigma \quad , \quad z_0 \text{ given} \\
& \|u\| \leq \sigma \\
& \sigma \leq \rho e^{-\tilde{z}} - \rho e^{-\tilde{z}}(z - \tilde{z}) \\
& \ln(m_0 - \alpha \rho t) \leq z \\
& \tilde{z}(t) = \begin{cases} \ln(m_0 - \alpha \rho t) , & m_0 - \alpha \rho t \geq m_{dry} \\ \ln(m_{dry}) , & \text{otherwise} \end{cases}
\end{aligned}$$

The above problem is probably impossible to solve indirectly (using the optimality conditions). However, it is easily discretized and solved directly (using Yalmip for example).

This analysis was based on the 2007 paper by Acikmese + Ploen called "Convex Programming Approach to Powered Descent Guidance for Mars Landing" in JGCD.

The previous problem originally had a control constraint of the form

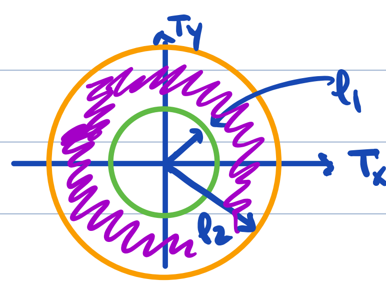
$$\|T\| \leq \rho$$

which meant the thrust magnitude was bounded. In such a case, the engine is allowed to turn off since this corresponds to $\|T\| = 0 < \rho$.

Having an engine turn off during descent is less than desirable since chemical thrusters have limited throttling capability — and once an engine is off it might not turn back on!

We can impose a throttling constraint as

$$\rho_1 \leq \|T\| \leq \rho_2$$

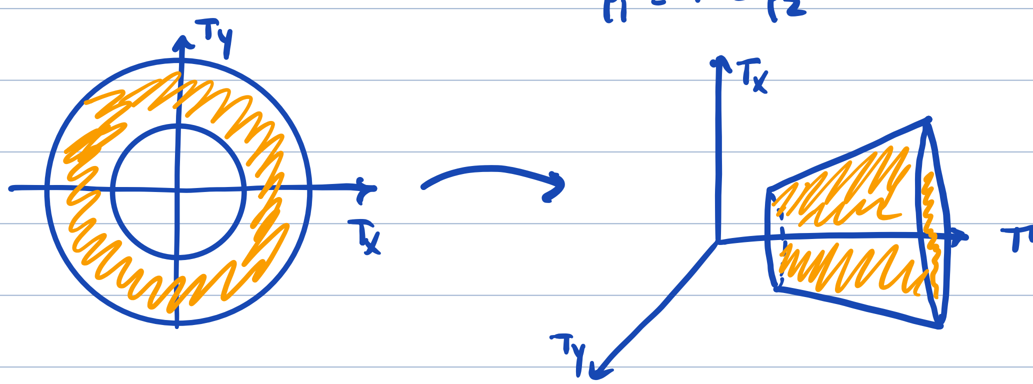


This type of constraint looks like a donut or annulus. Thus, it is non-convex and complicates our previous analysis.

We will now present a relaxation strategy for this constraint, i.e., a way to make this constraint convex.

To do this, we will use the following lifting (to an extra dimension) & relaxation (loosening of the constraints).

$$\rho_1 \le \|T\| \le \rho_2 \longrightarrow \begin{array}{l} \|T\| \le \Gamma \\ \rho_1 \le \Gamma \le \rho_2 \end{array}$$



We now reformulate our control problem as

$$\min \int_0^{t_f} \Gamma \, dt$$

$$\begin{aligned} \text{s.t.} \quad & \dot{r} = v && , \; r_0 \text{ given}, \; r_f \text{ given} \\ & \dot{v} = T/m + g && , \; v_0 \text{ given}, \; v_f \text{ given} \\ & \dot{m} = -\alpha \Gamma && , \; m_0 \text{ given} \\ & \|T\| \le \Gamma \\ & \rho_1 \le \Gamma \le \rho_2 \end{aligned}$$

For this to be an "exact" relaxation, we need to show that $\|T\| = \Gamma$ at all times. To show this, let's look at the optimality conditions.

$$H = \lambda_0 \Gamma + \lambda_1^T v + \lambda_2^T (T/m + g) - \alpha \lambda_3 \Gamma$$

The costate and transversality conditions are

$$\dot{\lambda}_1 = 0 \quad , \quad \lambda_{1f} = v_1$$

$$\dot{\lambda}_2 = -\lambda_1 \quad , \quad \lambda_{2f} = v_2$$

$$\dot{\lambda}_3 = \frac{\lambda_2^T T}{m^2} \quad , \quad \lambda_{3f} = 0$$

$$H_f = 0.$$

We will now show that $\lambda_2 = 0$ cannot hold everywhere. Suppose that it does. Then $\lambda_1 = 0$ and $\lambda_3 = 0$ everywhere. Then $H_f = 0$ implies $\lambda_0 = 0$, which violates non-triviality. Thus, λ_2 cannot be zero everywhere.

This means that $\lambda_2 = -\lambda_1 t + a$ for some $a \neq 0$.

The pointwise minimum condition says that the optimal control T must satisfy

$$T = \underset{\|T\| \le T'}{\arg\min} \; \frac{\lambda_2^T}{m} T$$

Since λ_2 / m is non-zero (except possibly at one point), the optimal thrust is

$$T = \frac{-\lambda_2 / m}{\|\lambda_2 / m\|} T' \quad , \quad \text{i.e.,} \quad \|T\| = T'.$$

Note that when $\|T\| = T'$, the non-convex constraint $\rho_1 \le \|T\| \le \rho_2$ is satisfied! In summary, we've proved that this convex relaxation will solve the original non-convex problem.

The paper by Acikmese & Ploen goes on to show that

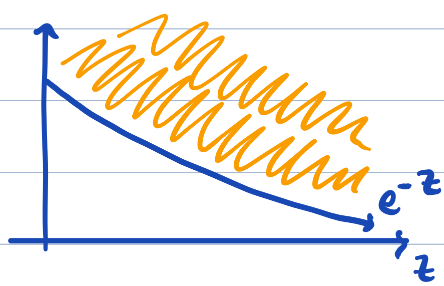
$$\|T\| = \rho_1 \quad \text{or} \quad \|T\| = \rho_2,$$

and it never takes an intermediate value. This is equivalent to showing singular arcs cannot happen.

After all this, the problem is still non-convex because of the non-linear dynamics. We will again need the u, σ, z transformations. Everything we did before holds but we now need to work on

$$\rho_1 \leq \|T\| \rightarrow \rho_1 \leq m\sigma \rightarrow \rho_1 e^{-z} \leq \sigma$$

This constraint is convex!



But it isn't a second-order cone constraint. To make it one, we'll use a second-order Taylor approximation about $\tilde{z}$. Then,

$$\rho_1 e^{-\tilde{z}} \left[1 - (z - \tilde{z}) + \frac{1}{2}(z - \tilde{z})^2 \right] \leq \sigma$$

And, using the mean value theorem, we can show that this is conservative.

The final convex problem is

$$\min \int_0^{t_f} \sigma(\tau)\, d\tau$$

$$\begin{aligned}
\text{s.t.} \quad & \dot{r} = v, && r_0 \text{ given}, && r_f \text{ given} \\
& \dot{v} = u + g, && v_0 \text{ given}, && v_f \text{ given} \\
& \dot{z} = -\alpha\sigma, && z_0 \text{ given} \\
& \|u\| \leq \sigma \\
& \sigma \leq \rho_2 e^{-\tilde{z}} - \rho_2 e^{-\tilde{z}}(z - \tilde{z}) \\
& \sigma \geq \rho_1 e^{-\tilde{z}}\left[1 - (z - \tilde{z}) + \tfrac{1}{2}(z - \tilde{z})^2\right] \\
& \ln(m_0 - \alpha\rho_2 t) \leq z \leq \ln(m_0 - \alpha\rho_1 t) \\
& \tilde{z}(t) = \begin{cases} \ln(m_0 - \alpha\rho_2 t), & m_0 - \alpha\rho_2 t \geq m_{dry} \\ \ln(m_{dry}), & \text{otherwise} \end{cases}
\end{aligned}$$

Like before, solving this indirectly is likely impossible. Discretizing it + solving it directly is easy. This problem can be solved onboard a flight computer with guaranteed convergence to global optimality in polynomial time.

And recall, all of this started w/ a non-convex problem!

Q-Guidance

In these notes, we'll explore a technique to transfer a spacecraft from one orbit to another using continuous (non-impulsive) thrust. It is called Q Guidance or cross-product steering. It is fuel optimal under a flat planet assumption and approximately so for a spherical planet. It was first developed for missiles and is currently planned for use on the 2nd stage of the Mars Ascent Vehicle (MAV).

As motivation, let's consider a problem on a flat planet with constant gravitational force. To reach a point $\bar{r}_1 := \bar{r}(t_1)$ by coasting from the point $\bar{r}(t)$, the velocity at this point $\bar{v}_r(t)$ must satisfy

$$\bar{r}(t_1) = \bar{r}(t) + (t_1 - t)\bar{v}_r(t) + \frac{1}{2}(t_1 - t)^2 \bar{g}$$

which comes from simple integration of $\dot{\bar{r}} = \bar{v}$, $\dot{\bar{v}} = \bar{g}$. Solving for $\bar{v}_r$ gives

$$\Rightarrow \quad \bar{v}_r(t) = \frac{1}{t_1 - t}\left[\bar{r}_1 - \bar{r}(t) - \frac{1}{2}(t_1 - t)^2 \bar{g}\right]$$

If, at this moment, the vehicle's velocity $\bar{v}(t)$ is not equal to $\bar{v}_r(t)$, i.e., the vehicle is not on a trajectory that will coast to the desired final point, then we must thrust to get there.

Denote the velocity-to-be-gained as $\bar{v}_g$ such that

$$\bar{v}_g = \bar{v}_r - \bar{v}$$

Differentiating and substituting

$$\Rightarrow \quad (t_1 - t)\dot{\bar{v}}_r - \bar{v}_r = -\bar{v} + (t_1 - t)\bar{g}$$

and $\dot{\bar{v}} = \bar{g} + \bar{a}_T \longleftarrow$ the thrust acceleration

$$\Rightarrow \quad \dot{\bar{v}}_g = \frac{1}{t_1 - t}\bar{v}_g - \bar{a}_T$$

we sometimes define $Q := \frac{-1}{t_1 - t} I$ such that

$$\dot{\bar{v}}_g = -Q\bar{v}_g - \bar{a}_T$$

hence the name Q guidance. To reach the desired trajectory, we must choose $\bar{a}_T$ to drive $\bar{v}_g \to 0$. To explore this further, see that

$$\frac{d}{dt}(\bar{v}_g \cdot \bar{v}_g) = \frac{d}{dt}(v_g^2) = 2\dot{\bar{v}}_g \cdot \bar{v}_g$$

$$= \frac{2}{t_1 - t} v_g^2 - 2\bar{a}_T \cdot \bar{v}_g$$

Since v_g^2 is non-negative, its derivative must be made negative to drive it to zero. To do so as quickly as possible we must choose $\bar{a}_T$ parallel to $\bar{v}_g$ and as large as possible in magnitude. If $T(t)$ is the upper bound on acceleration at time t,

$$\bar{a}_T(t) = \frac{\bar{v}_g(t)}{\|\bar{v}_g(t)\|} T(t).$$

Note that it may not be possible to make the derivative negative if T is not sufficiently large.

Observe that $Q \sim I$ for this problem, which is a very special situation. Also observe that this choice of $\bar{a}_T$ causes $\dot{\bar{v}}_g \times \bar{v}_g$ to be zero since

$$\begin{aligned}\dot{\bar{v}}_g \times \bar{v}_g &= \left(\frac{1}{t_1 - t}\bar{v}_g - \bar{a}_T\right) \times \bar{v}_g \\ &= \frac{1}{t_1 - t}\bar{v}_g \times \bar{v}_g - \bar{a}_T \times \bar{v}_g \\ &= 0\end{aligned}$$

According to Battin, it is this cross-product property that is important + generalizes to spherical bodies. Hence, Q guidance is also called cross-product steering.

Let's now explore the general case where

$$\dot{\bar{v}} = \bar{g}(\bar{r}) + \bar{a}_T$$

We again define

$$\bar{v}_g = \bar{v}_r - \bar{v}$$

s.t. $$\dot{\bar{v}}_g = \dot{\bar{v}}_r - \bar{g}(\bar{r}) - \bar{a}_T$$

Since $\bar{v}_r$ depends on t and $\bar{r}$, the chain rule gives

$$\frac{d\bar{v}_r}{dt} = \frac{\partial \bar{v}_r}{\partial t} + \frac{\partial \bar{v}_r}{\partial \bar{r}} \frac{d\bar{r}}{dt}$$

$$= \frac{\partial \bar{v}_r}{\partial t} + \frac{\partial \bar{v}_r}{\partial \bar{r}} \bar{v}$$

$$= \frac{\partial \bar{v}_r}{\partial t} + \frac{\partial \bar{v}_r}{\partial \bar{r}} (\bar{v}_r - \bar{v}_g)$$

$$= \underbrace{\frac{\partial \bar{v}_r}{\partial t} + \frac{\partial \bar{v}_r}{\partial \bar{r}} \bar{v}_r}_{= \frac{d\bar{v}_r}{dt} = \bar{g}(\bar{r}) \text{ since it is a coasting traj.}} - \frac{\partial \bar{v}_r}{\partial \bar{r}} \bar{v}_g$$

$$= \bar{g}(\bar{r}) - \frac{\partial \bar{v}_r}{\partial \bar{r}} \bar{v}_g$$

Substituting this back into the $\dot{\bar{v}}_g$ equation with $Q = \frac{\partial \bar{v}_r}{\partial \bar{r}}$ gives

$$\dot{\bar{v}}_g = -Q\bar{v}_g - \bar{a}_T$$

With this as our equation for $\dot{\bar{v}}_g$, Battin says we should choose $\bar{a}_T$ s.t. $\dot{\bar{v}}_g \times \bar{v}_g = 0$. Define

$$\bar{p}(t) = -Q(t)\bar{v}_g(t) \quad \text{s.t.} \quad \dot{\bar{v}}_g = \bar{p} - \bar{a}_T$$

Cross-product steering is then to choose $\bar{a}_T$ such that

$$(\bar{p} - \bar{a}_T) \times \bar{v}_g = \bar{p} \times \bar{v}_g - \bar{a}_T \times \bar{v}_g = 0$$

$$\Rightarrow \bar{a}_T \times \bar{v}_g = \bar{p} \times \bar{v}_g \qquad \text{(Does } \bar{a}_T \text{ have to equal } \bar{p}\text{?)}$$

where again the vehicle must have sufficient thrust to achieve this. Vector post-multiplication by $\bar{v}_g$ and using $(\bar{a}\times\bar{b})\times\bar{c} = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{b}\cdot\bar{c})\bar{a}$ yields

$$(\bar{a}_T \cdot \bar{v}_g)\bar{v}_g - v_g^2 \bar{a}_T = (\bar{p}\cdot\bar{v}_g)\bar{v}_g - v_g^2 \bar{p}$$

Dividing by v_g^2

$$\bar{a}_T = \bar{p} + \frac{1}{v_g^2}\left[\bar{a}_T \cdot \bar{v}_g - \bar{p}\cdot\bar{v}_g\right]\bar{v}_g$$

Denoting $\hat{i}_g = \bar{v}_g / v_g$ we can write

$$\bar{a}_T = \bar{p} + [\underbrace{\bar{a}_T \cdot \hat{i}_g}_{q} - \bar{p}\cdot\hat{i}_g]\,\hat{i}_g$$

$$= \bar{p} + (q - \bar{p}\cdot\hat{i}_g)\,\hat{i}_g \qquad (\Delta)$$

Squaring both sides gives

$$\bar{a}_T^T\bar{a}_T = \bar{p}^T\bar{p} + 2(q - \bar{p}\cdot\hat{i}_g)\,\bar{p}^T\hat{i}_g + (q - \bar{p}\cdot\hat{i}_g)^2$$

$$\Rightarrow \quad a_T^2 = p^2 + 2q\bar{p}\cdot\hat{i}_g - 2(\bar{p}\cdot\hat{i}_g)^2 + q^2 - 2q\bar{p}\cdot\hat{i}_g + (\bar{p}\cdot\hat{i}_g)^2$$

$$= p^2 + q^2 - (\bar{p}\cdot\hat{i}_g)^2$$

Using the above to solve for q gives

$$q = \left[a_T^2 - p^2 + (\bar{p}\cdot\hat{i}_g)^2\right]^{1/2} \qquad (\square)$$

From here, it is again evident that a_T must be sufficiently large for q to be real.

It is common to know the magnitude of thrust available at a given time. Thus, we use (□) to calculate q and then (Δ) to calculate $\bar{a}_T$. By doing this continuously, or periodically in guidance, $\bar{v}_q$ will be driven to zero. This approach <u>guides</u> us to the desired orbit.

When the available thrust is not sufficiently large, the thrust acceleration is chosen parallel to $\bar{v}_q$ and as large as possible in magnitude, i.e.,

$$\bar{a}_T = \frac{\bar{v}_q}{v_q} a_{T,max}$$

The calculation of $\bar{v}_r$ and Q depends on the target orbit and can be quite involved - depending on the situation.

Circularization

Consider a vehicle at position $\bar{r}$ with a goal of circularization in a possibly different plane defined by $\hat{i}_n$.
Then,

$$\bar{v}_r = \sqrt{\frac{\mu}{r}}\, \hat{i}_n \times \hat{i}_r$$

By driving $\bar{v}_g$ to zero, we control the shape (circular) and orientation ($\hat{i}_n$) but not the final radius.
By rewriting $\bar{v}_r$ as

$$\bar{v}_r = S_n \bar{r} \sqrt{\frac{\mu}{r^3}}$$

with
$$S_n = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$$

where n_x, n_y, and n_z are the direction cosines of $\hat{i}_n$
we find

$$Q = \sqrt{\frac{\mu}{r^3}}\, S_n \left(I - \frac{3}{2} \hat{i}_r \hat{i}_r^T \right)$$

Elliptic Target Orbit

To achieve an elliptic orbit with given p, e, and $\hat{i}_h$:

$$\bar{v}_r = \pm \left\{ \frac{\mu}{p}\left[e^2 - \left(\frac{p}{r}-1\right)^2\right] \right\}^{1/2} \hat{i}_r + \frac{\sqrt{\mu p}}{r}\, \hat{i}_h \times \hat{i}_r$$

$$Q = \pm \frac{\sqrt{\mu p}}{r^2}\left[\left(\frac{re}{p-r}\right)^2 - 1\right]^{-1/2} \hat{i}_r \hat{i}_r^T$$

$$\pm \frac{1}{r}\left\{ \frac{\mu}{p}\left[e^2 - \left(\frac{p}{r}-1\right)^2\right] \right\}^{1/2} \left(I - \hat{i}_r \hat{i}_r^T\right)$$

$$- \frac{\sqrt{\mu p}}{r^2} S_h \left(I - 2\hat{i}_r \hat{i}_r^T\right)$$

where S_h is the cross-product matrix w/ $S_h \hat{i}_r = \hat{i}_h \times \hat{i}_r$.

Let's try to derive $\bar{v}_r$ for the elliptic orbit insertion.

Given $\bar{r}$, p, e, and $\hat{i}_h$

1) Calculate $h = \sqrt{p\mu}$

2) Calculate v_r^2

$$1 - e^2 = p\left(\frac{2}{r} - \frac{v_r^2}{\mu}\right)$$

$$\Rightarrow \frac{1-e^2}{p} = \frac{2}{r} - \frac{v_r^2}{\mu}$$

$$\Rightarrow v_r^2 = \frac{2\mu}{r} - \frac{\mu}{p}(1-e^2)$$

* Note that the use of this equation assumes r is consistent w/ the desired orbit.

3) Calculate $\bar{v}_r$. Recognize that

$$\bar{v}_r \times \bar{r} = -\bar{h}$$

Postmultiplying by $\bar{r}$ gives

$$(\bar{v}_r \times \bar{r}) \times \bar{r} = -\bar{h} \times \bar{r}$$

$$\Rightarrow (\bar{v}_r \cdot \bar{r})\bar{r} - (\bar{r}\cdot\bar{r})\bar{v}_r = -\bar{h} \times \bar{r} = -h\hat{i}_h \times r\hat{i}_r$$

Solving for $\bar{v}_r$ gives

$$\bar{v}_r = \underbrace{(\bar{v}_r \cdot \hat{i}_r)}_{q} \hat{i}_r + \frac{\sqrt{p\mu}}{r} \hat{i}_n \times \hat{i}_r$$

Squaring both sides gives

$$v_r^2 = q^2 + \frac{p\mu}{r^2} + \underbrace{2q\frac{\sqrt{p\mu}}{r} \hat{i}_r^T (\hat{i}_n \times \hat{i}_r)}_{=0}$$

$$\Rightarrow \quad q = \pm\left[\frac{2\mu}{r} - \frac{\mu}{p}(1-e^2) - \frac{p\mu}{r^2}\right]^{1/2}$$

Hence,

$$\bar{v}_r = \pm\left[\frac{2\mu}{r} - \frac{\mu}{p}(1-e^2) - \frac{p\mu}{r^2}\right]^{1/2} \hat{i}_r + \frac{\sqrt{p\mu}}{r} \hat{i}_n \times \hat{i}_r$$

Is this the same as Battin? His $\hat{i}_r$ term is

$$\frac{\mu}{p}\left[e^2 - \left(\frac{p}{r} - 1\right)^2\right] = \frac{\mu}{p}\left[e^2 - \left(\frac{p^2}{r^2} - \frac{2p}{r} + 1\right)\right]$$

$$= \frac{\mu e^2}{p} - \frac{\mu p}{r^2} + \frac{2\mu}{r} - \frac{\mu}{p}$$

$$= \frac{2\mu}{r} - \frac{\mu}{p}(1-e^2) - \frac{p\mu}{r^2} \qquad \text{It is!}$$

Estimate of Burn Time

It is not uncommon for continuous burns to be of short duration, approximating impulsive burns. For example, the second stage burn of MAV is about 25 seconds. Assuming Q is constant (along w/ some other assumptions) we can derive an estimate for the burn time.

Because of cross-product steering, $\bar{v}_g$ is not rotating. Because Q is assumed constant, $\bar{p}$ will have a fixed direction proportional to $\bar{v}_g$. Let Av_g & Bv_g be the components of $\bar{p}$ along & perpendicular to $\bar{v}_g$. Then,

$$A = \frac{\bar{p}\cdot\bar{v}_g}{v_g^2}, \quad B = \left[\frac{\bar{p}\cdot\bar{p}}{v_g^2} - A^2\right]^{1/2}$$

Note that

$$\dot{\bar{v}}_g\cdot\dot{\bar{v}}_g = \left(\frac{dv_g}{dt}\right)^2 = (\bar{p}-\bar{a}_T)\cdot(\bar{p}-\bar{a}_T)$$

$$= p^2 + a_T^2 - 2\bar{p}\cdot a_T$$

Substituting in $\bar{a}_T = \bar{p} - \dot{\bar{v}}_g$ gives

$$\left(\frac{dv_g}{dt}\right)^2 = p^2 + a_T^2 - 2\bar{p}\cdot(\bar{p} - \dot{\bar{v}}_g)$$

$$= a_T^2 - p^2 + 2\bar{p}\cdot\dot{\bar{v}}_g$$

$$= a_T^2 - (A^2+B^2)v_g^2 + 2Av_g\frac{dv_g}{dt}$$

Solving for dv_g/dt using the quadratic formula & taking the negative root (since v_g should be decreasing)

$$\frac{dv_g}{dt} = -a_T\left[1 - \frac{B^2}{a_T^2}v_g\right]^{1/2} + Av_g$$

Expanding the root into a series and keeping only the first term gives

$$\frac{dv_g}{dt} = -a_T\left[1 - \frac{B^2}{a_T^2}v_g\right] + Av_g .$$

We now introduce a new variable y satisfying

$$\frac{1}{y}\dot{y} = -\frac{B^2}{2a_T}v_g$$

The resulting ODE for y is linear and 2^{nd}-order.

$$\ddot{y} + \underbrace{\left(\frac{1}{a_T}\dot{a}_T - A\right)}_{\text{time-varying}}\dot{y} - \frac{1}{2}B^2 y = 0$$

We now assume a constant thrust. Then

$$a_T = \frac{T}{m_0 - \dot{m}t} = \frac{T}{m_0}\left[1 + \frac{\dot{m}t}{m_0} + \left(\frac{\dot{m}}{m_0}\right)^2 t + \dots\right]$$

We now assume that the time rate of change of the thrust acceleration to the thrust acceleration is a constant. In other words, we assume the coefficient in the ODE is constant.

The solution to the ODE is then given by its characteristic values

$$2\lambda_1, 2\lambda_2 = -\frac{\dot{m}}{m_0} + A \pm \left[A^2 + 2B^2 + \frac{\dot{m}}{m_0}\left(\frac{\dot{m}}{m_0} - 2A\right)\right]^{1/2}$$

In terms of the original variable v_g (not y)

$$\frac{B^2}{2a_T} v_g = \frac{-\lambda_1 e^{\lambda_1 t} + c\lambda_2 e^{\lambda_2 t}}{e^{\lambda_1 t} + c e^{\lambda_2 t}}$$

where c can be resolved using the initial conditions. Defining

$$w = \frac{B^2 v_g(0)}{2a_T(0)} \quad \Rightarrow \quad c = -\frac{(w+\lambda_1)}{w+\lambda_2}$$

Finally, the burn time estimate t_b is found using the fact that $v_g(t_b) = 0$. Thus,

$$t_b \approx \frac{1}{\lambda_2 - \lambda_1} \ln\left[\frac{\lambda_1(\lambda_2+w)}{\lambda_2(\lambda_1+w)}\right].$$